Discrete Mathematics (2009 Spring)
Advanced Counting Techniques
(Chapter 7, 5 hours)

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§7.1 Recurrence Relations

Recurrence Relations

Definition

A *recurrence relation* (R.R., or just recurrence) for a sequence $\{a_n\}$ is an equation that expresses an in terms of one or more previous elements $a_0, \cdots, a_{n-1}$ of the sequence, for all $n \geq n_0$.

Definition

A particular sequence (described non-recursively) is said to solve the given recurrence relation if it is consistent with the definition of the recurrence.

- A given recurrence relation may have many solutions.

Example

Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \text{ for any } n \geq 2.$$

Which of the following are solutions?

- $a_n = 3n$: $2a_{n-1} - a_{n-2} = 2(3(n-1)) - (3(n-2)) = 3n$ (Yes)
- $a_n = 2^n$: $2a_{n-1} - a_{n-2} = 2(2^{n-1}) - (2^{n-2}) \neq 2^n$ (No)
- $a_n = 5$: $2a_{n-1} - a_{n-2} = 2(5) - (5) = 5$ (Yes)

Example Applications

Example

Recurrence relation for growth of a bank account with $P\%$ interest per given period:

$$M_n = M_{n-1} + (P/100)M_{n-1}.$$

Example

Growth of a population in which each organism yields 1 new one every period starting 2 periods after its birth.

$$P_n = P_{n-1} + P_{n-2} \text{ (Fibonacci relation).}$$

Solving Compound Interest RR

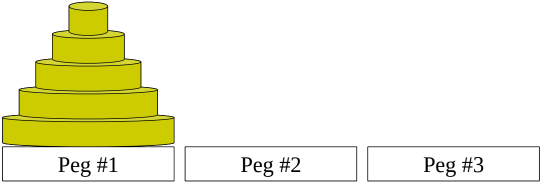
$$
\begin{aligned}
M_n &= M_{n-1} + (P/100)M_{n-1} \\
&= (1 + P/100)M_{n-1} \\
&= rM_{n-1} \qquad\qquad (\text{let } r = 1 + P/100) \\
&= r(rM_{n-2}) \\
&= r \cdot r \cdot (rMn - 3)) \qquad \ldots \text{ and so on to } \ldots \\
&= r^n M_0
\end{aligned}
$$

Tower of Hanoi Example

Problem

Get all disks from peg 1 to peg 2.

- *Only move 1 disk at a time.*
- *Never set a larger disk on a smaller one.*



Hanoi Recurrence Relation

Solution

Let $H_n = \#$ *moves for a stack of n disks.*
Optimal strategy:

- *Move top* $n-1$ *disks to spare peg.* *(*H_{n-1} *moves)*
- *Move bottom disk. (1 move)*
- *Move top* $n-1$ *to bottom disk.* *(*H_{n-1} *moves)*

Note: $H_n = 2H_{n-1} + 1.$

Solving Tower of Hanoi RR

Solution

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1} H_1 + 2^{n-2} + \ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \ldots + 2 + 1 \quad (since\ H_1 = 1) \\
&= \sum_{i=0}^{n-1} 2^i \\
&= 2^n - 1.
\end{aligned}
$$

Solving Recurrences

Definition

A linear homogeneous recurrence of degree k with constant coefficients is a recurrence of the form

$$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k},$$

where the c_i are all real, and $c_k \neq 0$.
The solution is uniquely determined if k initial conditions $a_0, a_1, \ldots, a_{k-1}$ are provided.

- What does linear means?
  - If s_n and t_n are solutions, for any real c and d, $cs_n + dt_n$ is solution, too.

Solving LiHoReCoCos

- Basic idea: Look for solutions of the form $a_n = r^n$, where r is a constant.
- Bring $a_n = r^n$ back to the recursive equation.

$$r^n = c_1 r^{n-1} + \cdots + c_k r^{n-k},$$
$$\text{i.e., } r^{n-k}(r^k - c_1 r^{k-1} - \cdots - c_k) = 0.$$

- The characteristic equation:

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0.$$

- The solutions (characteristic roots) can yield an explicit formula for the sequence.

Solving 2-LiHoReCoCos

- Consider an arbitrary 2-LiHoReCoCo:

$$a_n = c_1 a^{n-1} + c_2 a^{n-2}.$$

- It has the characteristic equation (C.E.):

$$r^2 - c_1 r - c_2 = 0.$$

Theorem

If this CE has 2 roots $r_1 \neq r_2$, then

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n \geq 0 \text{ for some constant } \alpha_1, \alpha_2.$$

Example

Solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ given the initial conditions $a_0 = 2$, and $a_1 = 7$.

Solution

- *Here* $c_1 = 1$ *and* $c_2 = 2$. *Then, the characteristic equation is* $r^2 - r - 2 = 0$. *We can get* $r = 2$ *or* $r = -1$. *So, assume*

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n.$$

- *To find* α_1 *and* α_2, *solve the equations by the initial conditions* $a_0 = 2$ *and* $a_1 = 7$:

$$\begin{array}{l} a0 = 2 = \alpha_1 2^0 + \alpha_2 (-1)^0 \\ a1 = 7 = \alpha_1 2^1 + \alpha_2 (-1)^1 \end{array} \Rightarrow \begin{array}{c} \alpha_1 + \alpha_2 = 2 \\ 2\alpha_1 - \alpha_2 = 7 \end{array}$$

We can get $\alpha_1 = 3$ *and* $\alpha_2 = -1$. *So,* $a_n = 3 \cdot 2^n - (-1)^n$.

Example

Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$ with $a_0 = 2$ and $a_1 = 1$ by characteristic equations.

Solution

Let $a_n = r^n$.

- *Then, the C.E. is $r^2 - 7r + 10 = 0$.*
- *There are two distinguish root $r = 2$ and $r = 5$. So, the two basic solution is 2^n and 5^n.*
- *Assume $a_n = \alpha_1 2^n + \alpha_2 5^n$. From $a_0 = 2$, we have $\alpha_1 + \alpha_2 = 2$. From $a_1 = 1$, we have $2\alpha_1 + 5\alpha_2 = 1$ Thus, we can get $\alpha_1 = 3$ and $\alpha_2 = -1$.*
- *So, $a_n = 3 \cdot 2^n - 5^n$.*

k-LiHoReCoCos

- Consider a k-LiHoReCoCo: $a_n = \sum_{i=1}^{k} c_i a_{n-i}$.
- It's C.E. is: $r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$.

Theorem

If this has k distinct roots r_i, then the solutions to the recurrence are of the form

$$a_n = \sum_{i=1}^{k} \alpha_i r_i^n$$

for all $n \geq 0$, where the α_i are constants.

Example

Solve $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$.

The Case of Degenerate Roots

Theorem

If the C.E. $r^2 - c_1 r - c_2 = 0$ *has only 1 root* r_0, *then*

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n,$$

for all $n \geq 0$, *for some constants* α_1 *and* α_2.

Proof.

Quiz! □

Example

Find all solutions to $a_n = 4a_{n-1} - 4a_{n-2}$ for $a_0 = 1$ and $a_1 = 0$?

Solution

- *The corresponding C.E. is* $r^2 - 4r + 4 = 0$. *So, we have a root* $r = 2$ *with a multiplicity* 2. *Therefore, the solution of the recurrence relation is* $a_n = \alpha_1 2^n + \alpha_2 n2^n$.
- *From* $a_0 = 1$, $\alpha_1 = 1$. *From* $a_1 = 0$, *we have* $2\alpha_1 + 2\alpha_2 = 0$. *So, we can get* $\alpha_1 = 1$ *and* $\alpha_2 = -1$. *Thus,*

$$a_n = 2^n - n2^n.$$

Degenerate k-LiHoReCoCos

Theorem

Suppose there are t roots $r_1, \cdots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then,

$$a_n = \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j \right) r_i^n$$

for all $n \geq 0$, where all the $\alpha_{i,j}$ are constants.

Example

1. Solve $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$.
2. Solve $a_n = 3a_{n-1} - 4a_{n-3}$.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (LiNoReCoCos)

Definition

Linear *nonhomogeneous* RRs with constant coefficients may (unlike LiHoReCoCos) contain some terms $F(n)$ that depend only on n (and not on any a_i's)

$$\underbrace{a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}}_{\substack{\text{The associated homogeneous} \\ \text{recurrence relation.} \\ \text{(associated LiHoReCoCo)}}} + F(n)$$

Solutions of LiNoReCoCos

Theorem

If $a_n = p(n)$ is any particular solution to the LiNoReCoCo

$$a_n = \left(\sum_{i=1}^{k} c_i a_{n-i}\right) + F(n),$$

then all its solutions are of the form

$$a_n = p(n) + h(n),$$

where $a_n = h(n)$ is any solution to the associated homogeneous RR

$$a_n = \sum_{i=1}^{k} c_i a_{n-i}.$$

Example

Find all solutions to $a_n = 3a_{n-1} + 2n$. Which solution has $a_1 = 3$?

Solution

- *The associated 1-LiHoReCoCo is* $h_n = 3h_{n-1}$*, whose solutions are all of the form* $h_n = \alpha 3^n$*. Thus the solutions to the original problem are all of the form* $a_n = \alpha 3^n + p_n$*. So, all we need to do is find one* p_n *that works.*
- *If the extra term* $F(n)$ *is a degree-t polynomial in n, you should try a degree-t polynomial as the particular solution* p_n*. In this case, try* $p_n = cn + d$ $(p_n = 3p_{n-1} + 2n)$*. Then,* $cn + d = 3(c(n-1) + d) + 2n$*, i.e.* $(-2c + 2)n + (3c - 2d) = 0$.

Solution ((Cont.))

Solve the system $\begin{cases} -2c+2=0 \\ 3c-2d=0 \end{cases}$. *Then, we have* $c=1$ *and* $d=3/2$. *So,*

$$p_n = n + \frac{3}{2}.$$

Now, we know that solutions to our example are of the form

$$a_n = h_n + p_n = \alpha 3^n + n + \frac{3}{2}.$$

Solve α *by the given case* $a_1 = 3$. *From* $3 = \alpha \cdot 3^1 + 1 - \frac{3}{2}$, *we have* $\alpha = 11/6$. *So, the answer is*

$$a_n = \frac{7}{6}3^n + n + \frac{3}{2}.$$

Example

Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2} + 16n + 5$ for $n \geq 2$ with $a_0 = 0$ and $a_1 = 4$ by characteristic equations.

Solution

Let $a_n = h_n + p_n$*. For the homogeneous part* $(h_n = 7h_{n-1} - 10h_{n-2})$*, we have* $h_n = \alpha_1 2^n + \alpha_2 5^n$*. Now, assume* $p_n = an + b$ *and* $p_n = 7p_{n-1} - 10p_{n-2} + 16n + 5$*. Then, we have*

$$(an + b) = 7\left(a\left(n - 1\right) + b\right) - 10\left(a\left(n - 2\right) + b\right) + 16n + 5.$$

This is a polynomial equation of n*.*

Solution (Cont.)

By comparson the coefficent, we can get $a = 4$ *and* $b = \frac{57}{4}$. *Thus,*

$$a_n = h_n + p_n = \alpha_1 2^n + \alpha_2 5^n + 4n + \frac{57}{4}.$$

From $a_0 = 0$, *we have* $\alpha_1 + \alpha_2 = -\frac{57}{4}$. *From* $a_1 = 1$, *we have* $2\alpha_1 + 5\alpha_2 = -\frac{69}{4}$. *Thus, we can get* $\alpha_1 = -18$ *and* $\alpha_2 = \frac{15}{4}$. *So,* $a_n = (-18)\,2^n + \frac{15}{4}5^n + 4n + \frac{57}{4}$.

Divide & Conquer R.R.s

- Many types of problems are solvable by reducing a problem of size n into some number a of independent subproblems, each of size $\leq \left\lceil \frac{n}{b} \right\rceil$, where $a \geq 1$ and $b > 1$.
- The time complexity to solve such problems is given by a recurrence relation

$$T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + g(n).$$

Divide & Conquer Examples

Example (Binary search)

Break list into 1 sub-problem (smaller list) (so $a = 1$) of size $\leq \left\lceil \frac{n}{2} \right\rceil$ (so $b = 2$). So,

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + c \qquad (g(n) = c \text{ constant}).$$

Example (Merge sort)

Break list of length n into 2 sublists ($a = 2$), each of size $\leq \left\lceil \frac{n}{2} \right\rceil$ (so $b = 2$), then merge them, in $g(n) = \Theta(n)$ time. So

$$T(n) = 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \qquad (\text{roughly for some } c).$$

Fast Multiplication Example

- The ordinary grade-school algorithm takes $\Theta\left(n^2\right)$ steps to multiply two n-digit numbers.
  - This seems like too much work!
- So, let's find an asymptotically faster multiplication algorithm!
- To find the product cd of two $2n$-digit base-b numbers, $c = \left(c_{2n-1}c_{2n-2}\cdots c_0\right)_b$ and $d = \left(d_{2n-1}d_{2n-2}\ldots d_0\right)_b$, first, we break c and d in half

$$\begin{aligned} c &= b^n C_1 + C_0, \\ d &= b^n D_1 + D_0. \end{aligned}$$

and then...

Derivation of Fast Multiplication

$$
\begin{aligned}
cd &= (b^n C_1 + C_0)(b^n D_1 + D_0) \\
&= b^{2n} C_1 D_1 + b^n (C_1 D_0 + C_0 D_1) + C_0 D_0 \\
&= b^{2n} C_1 D_1 + C_0 D_0 \\
&\quad + b^n (C_1 D_0 + C_0 D_1 + (C_1 D_1 - C_1 D_1) + (C_0 D_0 - C_0 D_0)) \\
&= (b^{2n} + b^n) C_1 D_1 + (b^n + 1) C_0 D_0 \\
&\quad + b^n (C_1 D_0 - C_1 D_1 - C_0 D_0 + C_0 D_1) \\
&= (b^{2n} + b^n) C_1 D_1 + (b^n + 1) C_0 D_0 \\
&\quad + b^n (C_1 - C_0)(D_0 - D_1)
\end{aligned}
$$

Recurrence Rel. for Fast Mult.

Notice that the time complexity $T(n)$ of the fast multiplication algorithm obeys the recurrence

$$T(2n) = 3T(n) + \Theta(n).$$

In other words,

$$T(n) = 3T(\frac{n}{2}) + \Theta(n)$$

So $a = 3$ and $b = 2$.

The Master Theorem

Theorem

Consider a function $f(n)$ that, for all $n = b^k$ for all $k \in \mathbb{Z}^+$, satisfies the recurrence relation

$$f(n) = af\left(\frac{n}{b}\right) + cn^d$$

with $a \geq 1$, integer $b > 1$, real $c > 0, d \geq 0$. Then,

$$f(n) \in \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log n\right) & \text{if } a = b^d \\ O\left(n^{\log_b a}\right) & \text{if } a > b^d \end{cases}$$

Example (The master theorem)

Recall that complexity of fast multiply was

$$T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n).$$

Thus, $a = 3$, $b = 2$, and $d = 1$. Since $a > b^d$, case 3 of the master theorem applies. So,

$$T(n) = O\left(n^{\log_b a}\right) = O\left(n^{\log_2 3}\right)$$

which is $O\left(n^{1.58\ldots}\right)$.
The new algorithm is strictly faster than ordinary $\Theta\left(n^2\right)$ multiply!

Generating Functions

Definition

The (ordinary) generating function for the sequence $a_0, a_1, \cdots, a_k, \cdots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{i=0}^{\infty} a_i x^i.$$

Example

1. If $a_i = 1$ for $i = 0, 1, \cdots$, $G(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}$.
2. If $a_i = \frac{1}{i!}$ for $i = 0, 1, \cdots$, $G(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots = e$.

Counting Problem 1

Example

Find the number of solutions of $e_1 + e_2 + e_3 = 17$.

Solution

The answer is equal to the coefficient of x^{17} *in the expansion of*

$$(1 + x + x^2 + \dots)(1 + x + x^2 + \dots)(1 + x + x^2 + \dots).$$

Example

What is the answer if $2 \leq x_1 \leq 8$, $3 \leq x_2 \leq 9$, and $4 \leq x_3 \leq 10$.

Solution

The answer is equal to the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + \cdots + x^8)(x^3 + x^4 + \cdots + x^9)(x^4 + x^5 + \cdots + x^{10}).$$

Problem

How to handle other more general cases, e.g. even numbers, odd numbers?

Counting Problem 2

Example

How many ways to pay r dollars into a vending machine with tokens worth \$1, \$2, and \$5.

Solution

If the order in which the tokens are inserted doesn't matter, the answer is given by the coefficient of x^r in the generating function

$$\begin{aligned} G(x) &= \left(1+x+x^2+\cdots\right)\left(1+x^2+x^4+\cdots\right) \\ &\quad \left(1+x^5+x^{10}+\cdots\right). \end{aligned}$$

Solution ((Cont.))

If the order in which the tokens are inserted matters and exactly n tokens are used, the answer is the coefficient of x^r *in the generating function*

$$G(x) = \left(x + x^2 + x^5\right)^n.$$

If the order in which the tokens are inserted matters, the answer is the coefficient of x^r *in the generating function*

$$\begin{aligned} G(x) &= 1 + \left(x + x^2 + x^5\right) + \left(x + x^2 + x^5\right)^2 + \cdots \\ &= \frac{1}{1 - \left(x + x^2 + x^5\right)}. \end{aligned}$$

Extended Binomial Theorem

Definition

Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient is defined by

$$\binom{u}{k} = \begin{cases} \frac{u(u-1)\cdots(u-k+1)}{k!} & \text{if } k > 0; \\ 1 & \text{if } k = 0. \end{cases}$$

Theorem

Let x be a real number with $|x| < 1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

This can be explained by the Taylor expansion.

Useful Generating Functions

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)\, x^k = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Solve Recurrence Relation: Example (1)

Example

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, \cdots$ and initial condition $a_0 = 2$.

Solution

Let $G(x)$ *be the generating function for the sequence* $\{a_n\}$. *Then*

$$\begin{aligned} G(x) &= \sum_{k=0}^{\infty} a_k x^k = \left(\sum_{k=1}^{\infty} 3a_{k-1} x^k \right) + a_0 \\ &= 3x \left(\sum_{k=1}^{\infty} a_{k-1} x^{k-1} \right) + 2 \\ &= 3xG(x) + 2. \end{aligned}$$

Solution ((Cont.))

Thus, we have

$$G(x) = \frac{2}{1-3x} = 2\sum_{k=0}^{\infty}(3x)^k = \sum_{k=0}^{\infty}\left(2\cdot 3^k\right)x^k.$$

So,

$$a_k = 2\cdot 3^k.$$

Solve Recurrence Relation: Example (2)

Example

Solve the recurrence relation $a_n = 8a_{n-1} + 10^{n-1}$ with the initial condition $a_1 = 9$.

Solution

From $a_1 = 8a_0 + 10^0$ and $a_1 = 9$, we have $a_0 = 1$. Let $G(x)$ be the generating function w.r.t. $\{a_n\}$.

$$\begin{aligned} G(x) &= \sum_{k=0}^{\infty} a_k x^k = \left(\sum_{k=1}^{\infty} \left(8a_{k-1} + 10^{k-1}\right) x^k \right) + a_0 \\ &= 8x \sum_{k=1}^{\infty} a_{k-1} x^{k-1} + x \sum_{k=1}^{\infty} (10x)^{k-1} + 1 \\ &= 8xG(x) + \frac{x}{1-10x} + 1. \end{aligned}$$

Solution ((Cont.))

So, we have

$$\begin{aligned} G(x) &= \frac{1-9x}{(1-8x)(1-10x)} = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right) \\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty} 8^k x^k + \sum_{k=0}^{\infty} 10^k x^k\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{2}\left(8^k + 10^k\right) x^k. \end{aligned}$$

Therefore,

$$a_n = \frac{1}{2}\left(8^k + 10^k\right).$$

Example

Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$ with $a_0 = 2$ and $a_1 = 1$ by generating functions.

Solution

First, find the close form of the generating functions.

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=2}^{\infty} a_n x^n \right) + a_1 x + a_0 \\
&= \left[\sum_{n=2}^{\infty} (7a_{n-1} - 10a_{n-2}) x^n \right] + x + 2 \\
&= 7x \left(\sum_{n=2}^{\infty} a_{n-1} x^{n-1} \right) - 10x^2 \left(\sum_{n=2}^{\infty} a_{n-2} x^{n-2} \right) + x + 2 \\
&= 7x (G(x) - a_0) - 10x^2 G(x) + x + 2 \\
&= 7xG(x) - 10x^2 G(x) - 13x + 2.
\end{aligned}$$

Solution ((Cont.))

So, we have $G(x)\left(10x^2-7x+1\right)=-13x+2.$

$$\begin{aligned} G(x) &= \frac{-13x+2}{10x^2-7x+1}=\frac{-13x+2}{(1-2x)(1-5x)} \\ &= \frac{a}{(1-2x)}+\frac{b}{(1-5x)}. \end{aligned}$$

Then, we can solve $a=3$ *and* $b=-1$.

$$\begin{aligned} G(x) &= \frac{3}{(1-2x)}-\frac{1}{(1-5x)}=3\sum_{n=0}^{\infty}(2x)^n-\sum_{n=0}^{\infty}(5x)^n \\ &= \sum_{n=0}^{\infty}\left(3\cdot 2^n-5^n\right)2x^n. \end{aligned}$$

Thus, $a_n=3\cdot 2^n-5^n$.

Example

Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2^n$ for $n \geq 2$ with $a_0 = 4$ and $a_1 = 12$ by generating function.

Solution

Find the close form of the generating function.

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=2}^{\infty} a_n x^n\right) + a_1 x + a_0 \\
&= \left[\sum_{n=2}^{\infty} \left(a_{n-1} + 2a_{n-2} + 2^n\right) x^n\right] + 12x + 4 \\
&= x\left(\sum_{n=2}^{\infty} a_{n-1} x^{n-1}\right) + 2x^2 \left(\sum_{n=2}^{\infty} a_{n-2} x^{n-2}\right) \\
&\quad + \left(\sum_{n=2}^{\infty} (2x)^n\right) + 12x + 4 \\
&= x\left(G(x) - a_0\right) + 2x^2 G(x) + \frac{4x^2}{1-2x} + 12x + 4 \\
&= xG(x) + 2x^2 G(x) + 8x + 4.
\end{aligned}$$

Solution ((Cont.))

So, we have

$$G\left(x\right)\left(-2x^2-x+1\right)=\frac{4x^2}{1-2x}-13x+2=\frac{-12x^2+4}{1-2x},$$

and

$$\begin{aligned} G\left(x\right) &= \frac{-12x^2+4}{\left(1-2x\right)^2\left(1+x\right)} \\ &= \frac{a}{\left(1-2x\right)^2}+\frac{b}{\left(1-2x\right)}+\frac{c}{\left(1+x\right)} \\ &= \frac{\left(-2b+4c\right)x^2+\left(a-b-4c\right)x+\left(a+b+c\right)}{\left(1-2x\right)^2\left(1+x\right)}. \end{aligned}$$

Solution ((Cont.))

Then, we can solve $a = \frac{6}{9}$, $b = \frac{38}{9}$, *and* $c = -\frac{8}{9}$.

$$\begin{aligned} & G(x) \\ = \; & \frac{6}{9}\frac{1}{(1-2x)^2} + \frac{38}{9}\frac{1}{(1-2x)} - \frac{8}{9}\frac{1}{(1+x)} \\ = \; & \left(\sum_{n=0}^{\infty} \frac{6}{9}(n+1)\,2^n x^n\right) + \left(\sum_{n=0}^{\infty} \frac{38}{9} 2^n x^n\right) - \left(\sum_{n=0}^{\infty} \frac{8}{9}(-1)^n x^n\right) \\ = \; & \sum_{n=0}^{\infty} \left(\frac{6}{9}(n+1)\,2^n + \frac{38}{9}2^n + \frac{8}{9}(-1)^n\right) x^n. \end{aligned}$$

Thus, $a_n = \frac{6}{9}(n+1)\,2^n + \frac{38}{9}2^n + \frac{8}{9}(-1)^n$.

Theorem

For finite sets $A_1, A_2, \cdots, A_n$,

$$\left|\bigcup_{1\le i\le n} A_i\right| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots .$$

Especially, if A *and* B *are sets, then* $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof.

The proof is given later. Please go down three pages to find the proof outline. □

Example

How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution

Let A (and B, respectively) be a set of integers not exceeding 1000 are divisible by 7 (and 11, respectively). Then,

- $|A| = \lfloor \frac{1000}{7} \rfloor = 142$, $|B| = \lfloor \frac{1000}{11} \rfloor = 90$, *and* $|A \cap B| = \lfloor \frac{1000}{77} \rfloor = 12$.
- $|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$.

Example

A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Solution

Let S, F, and R be sets of students who have taken a course in Spanish, French, and Russian, respectively.

- $|S| = 1232$, $|F| = 879$, $|R| = 114$.
- $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$.
- $|S \cap F \cap R| = 2092$.

So, from

$$\begin{aligned} |S \cup F \cup R| &= |S| + |F| + |R| \\ &\quad - |S \cap F| - |S \cap R| - |F \cap R| \\ &\quad + |S \cap F \cap R|, \end{aligned}$$

we have
$|S \cap F \cap R| = 2092 - 1232 - 879 - 114 + 103 + 23 + 14 = 7.$

Proof of the inclusion-exclusion principle.

Prove by mathematical induction.

$$
\begin{aligned}
& |A_1 \cup A_2 \cup \cdots \cup (A_n \cup A_{n+1})| \\
= \quad & \left(\left(\sum_{1 \le i \le n-1} |A_i| \right) + |A_n \cup A_{n+1}| \right) \\
& - \left(\begin{array}{c} \left(\sum_{1 \le i < j \le n-1} |A_i \cap A_j| \right) \\ + \left(\sum_{1 \le i \le n-1} |A_i \cap (A_n \cup A_{n+1})| \right) \end{array} \right) \\
& + \left(\begin{array}{c} \left(\sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k| \right) \\ + \left(\sum_{1 \le i < j \le n-1} |A_i \cap A_j \cap (A_n \cup A_{n+1})| \right) \end{array} \right) \\
& - \cdots
\end{aligned}
$$

(Cont.)

$$
\begin{aligned}
= \; & \left(\left(\sum_{1 \le i \le n-1} |A_i| \right) + |A_n| + |A_{n+1}| - |A_n \cap A_{n+1}| \right) \\
& - \left(\begin{array}{c} \left(\sum_{1 \le i < j \le n-1} |A_i \cap A_j| \right) \\ + \left(\sum_{1 \le i \le n-1} |(A_i \cap A_n) \cup (A_i \cap A_{n+1})| \right) \end{array} \right) \\
& + \left(\begin{array}{c} \left(\sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k| \right) \\ + \left(\sum_{1 \le i < j \le n-1} |(A_i \cap A_j \cap A_n) \cup (A_i \cap A_j \cap A_{n+1})| \right) \end{array} \right) \\
& - \cdots
\end{aligned}
$$

(Cont.)

$$
\begin{aligned}
= \; & \left(\left(\sum_{1 \le i \le n-1} |A_i| \right) + |A_n| + |A_{n+1}| - |A_n \cap A_{n+1}| \right) \\
& - \left(\begin{array}{c} \left(\displaystyle\sum_{1 \le i < j \le n-1} |A_i \cap A_j| \right) \\ + \left(\displaystyle\sum_{1 \le i \le n-1} \begin{array}{c} |A_i \cap A_n| + |A_i \cap A_{n+1}| \\ - |A_i \cap A_n \cap A_{n+1}| \end{array} \right) \end{array} \right) \\
& + \left(\begin{array}{c} \left(\displaystyle\sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k| \right) \\ + \left(\displaystyle\sum_{1 \le i < j \le n-1} \begin{array}{c} |A_i \cap A_j \cap A_n| + |A_i \cap A_j \cap A_{n+1}| \\ - |A_i \cap A_j \cap A_n \cap A_{n+1}| \end{array} \right) \end{array} \right) \\
& - \cdots
\end{aligned}
$$

(Cont.)

$$
\begin{aligned}
= & \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| \\
& + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots .
\end{aligned}
$$

□

Example

How many onto functions are there from a set with six elements to a set with three elements.

Proof.

Assume the elements in the codomain are b_1, b_2, b_3. Let U be the set of all possible functions and P_i for $i = 1, 2, 3$ be the set of functions that does not map to b_i. Then,

$$\begin{aligned}
\left|\overline{P_1 \cup P_2 \cup P_3}\right| &= |U - (P_1 \cup P_2 \cup P_3)| \\
&= |U| - \left[\begin{array}{c} |P_1| + |P_2| + |P_3| \\ -(|P_1 \cap P_2| + |P_1 \cap P_3| \cap |P_2 \cap P_3|) \\ + |P_1 \cap P_2 \cap P_3| \end{array}\right] \\
&= 3^6 - \binom{3}{1} 2^6 + \binom{3}{2} 1^6 + \binom{3}{3} 0^6 \\
&= 540.
\end{aligned}$$

Derangement

Definition

'A derangement is a permutation of objects that leaves no object in its original position. For example, 23154 is a dearangemetn of 34521.

Theorem

The number of derangement of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$