Discrete Mathematics

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§2.4 Sequences and Summations

($\sim$ 1.5 hours)

Sequences

- A *sequence* or *series* $\{a_n\}$ is identified with a *generating function* $f : S \longrightarrow A$ for some subset $S \subseteq \mathbb{N}$ (often $S = \{0, 1, 2, \ldots\}$ or $S = \{1, 2, 3, \ldots\}$) and for some set A.
- If f is a generating function for a series $\{a_n\}$, then for $n \in S$, the symbol a_n denotes $f(n)$, also called *term* n of the sequence.
- The *index* of a_n is n. (Or, often i is used.)
- Many sources just write "the sequence $a_1, a_2, \ldots$" instead of $\{a_n\}$, to ensure that the set of indices is clear.

Example (Infinite Sequences)

- Consider the series $\{a_n\} = a_1, a_2, \cdots$ where $(\forall n \geq 1)$ $a_n = f(n) = \frac{1}{n}$. Then,

$$\{a_n\} = 1, \frac{1}{2}, \frac{1}{3}, \cdots .$$

- Consider the sequence $\{b_n\} = b_0, b_1, \cdots$ (note 0 is an index) where $b_n = (-1)^n$. Then,

$$\{b_n\} = 1, -1, 1, -1, \cdots .$$

  Note repetitions! $\{b_n\}$ denotes an infinite sequence of 1's and -1's, not the 2-element set $\{1, -1\}$.

Example (Geometric progression)

$a, ar, ar^2, \cdots, ar^n, \cdots$

- $a_n = ar^{n-1}$
  - a is the initial term.
  - r is the common ratio.
- Ex: $3, -6, 12, \cdots, 3 \cdot (-2)^{n-1}, \cdots$

Example (Arithmetic progression)

$a, a+d, a+2d, \cdots, a+nd, \cdots$

- $a_n = a + (n-1)d$
  - a is the initial term.
  - d is the common difference.
- Ex: $4, 7, 10, \cdots, 4 + (n-1) \cdot 3, \cdots$

Recognizing Sequences

Example (What's the next number?)

- $1, 2, 3, 4, \ldots$ 5 (the 5th smallest number > 0)
- $1, 3, 5, 7, 9, \ldots$ 11 (the 6th smallest odd number > 0)
- $2, 3, 5, 7, 11, \ldots 13$ (the 6th smallest prime number)

- Sometimes, you're given the first few terms of a sequence, and you are asked to find the sequence's generating function, or a procedure to enumerate the sequence.
- The trouble with recognition
  - The problem of finding "the" generating function given just an initial subsequence is *not well defined*. There are *infinitely* many computable functions that will generate any given initial subsequence. (Prove this!)

Summation Notation

- Given a series $\{a_n\}$, an integer *lower bound*(*or limit*) $j \geq 0$, and an integer *upper bound* $k \geq j$, then the *summation of* $\{a_n\}$ *from* j *to* k is written and defined as follows

$$\sum_{i=j}^{k} a_i = a_j + a_{j+1} + \cdots + a_k.$$

E.g.,

$$\begin{aligned}\sum_{i=2}^{4} i^2 + 1 &= (2^2+1) + (3^2+1) + (4^2+1) \\ &= (4+1) + (9+1) + (16+1) \\ &= 5 + 10 + 17 \\ &= 32.\end{aligned}$$

Here, i is called the *index of summation*.

Generalized Summations

- For an infinite series, we may write

$$\sum_{i=j}^{\infty} a_i = a_j + a_{j+1} + \cdots .$$

- To sum a function over all members of a set $X = \{x_1, x_2, \ldots\}$:

$$\sum_{x \in X} f(x) = f(x_1) + f(x_2) + \cdots .$$

- Or, if $X = \{x | P(x)\}$, we may just write:

$$\sum_{P(x)} f(x) = f(x_1) + f(x_2) + \cdots .$$

More Summation Examples

Example

An infinite series with a finite sum

$$\sum_{i=0}^{\infty} 2^{-i} = 2^0 + 2^{-1} + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2.$$

Example

Using a predicate to define a set of elements to sum over

$$\sum_{(x \text{ is prime}) \wedge (x<10)} x^2 = 2^2 + 3^2 + 5^2 + 7^2 = 4 + 9 + 25 + 49 = 87.$$

Summation Manipulations

- Some handy identities for summations:
  - Distributive law

$$\sum_x cf(x) = c\sum_x f(x).$$

  - Application of commutativity

$$\sum_x f(x) + g(x) = \left(\sum_x f(x)\right) + \left(\sum_x g(x)\right).$$

  - Index shifting

$$\sum_{i=j}^{k} f(i) = \sum_{i=j+n}^{k+n} f(i-n).$$

- More summation manipulations
  - Series splitting

$$\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{m} f(i)\right) + \left(\sum_{i=m+1}^{k} f(i)\right) \text{ if } j \leq m < k.$$

  - Order reversal

$$\sum_{i=j}^{k} f(i) = \sum_{i=0}^{k-j} f(k-i).$$

  - Grouping

$$\sum_{i=0}^{2k} f(i) = \sum_{i=0}^{k} f(2i) + f(2i+1).$$

Example (Euler's Trick)

Evaluate the summation $\sum_{i=1}^{n} i$

Solution

There is a simple closed-form formula for the result, discovered by Euler at age 12!

- *Consider the sum*

$$\begin{aligned} & 1+2+\cdots+\frac{n}{2}+(\frac{n}{2}+1)+\cdots+(n-1)+n \\ = \quad & (n+1)+(n+1)+\cdots+(n+1). \end{aligned}$$

- $\frac{n}{2}$ *pairs of elements, each pair summing to* $n+1$*, for a total of* $\frac{n}{2}(n+1)$.

Geometric Progression

- A *geometric progression* is a series of the form $a, ar, ar^2, ar^3, \ldots, ar^k$, where $a, r \in \mathbb{R}$.
- The sum of such a series is given by:

$$S = \sum_{i=0}^{k} ar^i$$

- We can reduce this to *closed form* via clever manipulation of summations...

Geometric Sum Derivation

From $S = \sum_{i=0}^{n} ar^i$,

$$\begin{aligned} rS &= r\sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} ar^{i+1} = \sum_{i=1}^{n+1} ar^i = \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} \\ &= \left(\sum_{i=0}^{n} ar^i\right) + \left(ar^{n+1} - a\right) = S + \left(ar^{n+1} - a\right). \end{aligned}$$

So,

$$rS - S = a(r^{n+1} - 1),$$

and we have

$$S = \frac{a(r^{n+1} - 1)}{r - 1}.$$

Nested Summations

Example

Find $\sum_{i=1}^{4}\sum_{j=1}^{3} ij$.

Solution

$$\begin{aligned}\sum_{i=1}^{4}\sum_{j=1}^{3} ij &= \sum_{i=1}^{4}\left(\sum_{j=1}^{3} ij\right) = \sum_{i=1}^{4} i\left(\sum_{j=1}^{3} j\right) = \sum_{i=1}^{4} i(1+2+3) \\ &= \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6(1+2+3+4) \\ &= 6\cdot 10 = 60.\end{aligned}$$

Some Shortcut Expressions

$\sum_{k=0}^{n} ar^k = \frac{a(r^{n+1}-1)}{r-1}, r \neq 1$	Geometric series.
$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$	Euler's trick.
$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$	Quadratic series.
$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$	Cubic series.
$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $\lvert x \rvert < 1$	The Taylor series of $\frac{1}{1-x}$.
$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$, for $\lvert x \rvert < 1$	The Taylor series of $\frac{1}{(1-x)^2}$.

Example

Evaluate $\sum_{k=50}^{100} k^2$.

Solution

We have $\sum_{k=1}^{100} k^2 = \left(\sum_{k=1}^{49} k^2\right) + \sum_{k=50}^{100} k^2$. *So,*

$$\begin{aligned} \sum_{k=50}^{100} k^2 &= \left(\sum_{k=1}^{100} k^2\right) - \sum_{k=1}^{49} k^2 \\ &= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} \\ &= 338350 - 40425 \\ &= 297925. \end{aligned}$$

Cardinality: Formal Definition

Definition

For any two (possibly infinite) sets A and B, we say that A and B have the same cardinality(written $|A| = |B|$) iff there exists a bijection(bijective function) from A to B.

- When A and B are finite, it is easy to see that such a function exists iff A and B have the same number of elements $n \in \mathbb{N}$

Countable versus Uncountable

- For any set S, if S is finite or $|S| = |\mathbb{N}|$, we say S is *countable*. Else, S is *uncountable*.
- Intuition behind "**countable**:" we can enumerate (generate in series) elements of S in such a way that *any* individual element of S will eventually be *counted* in the enumeration.
  - E.g., $\mathbb{N}$, $\mathbb{Z}$.
- **Uncountable**: No series of elements of S (even an infinite series) can include all of S's elements.
  - E.g., $\mathbb{R}$, $\mathbb{R}^2$, $P(\mathbb{N})$.

Countable Sets: Examples

Theorem

The set $\mathbb{Z}$ is countable.

Proof.

Consider $f : \mathbb{Z} \to \mathbb{N}$ where $f(i) = 2i$ for $i \geq 0$ and $f(i) = -2i - 1$ for $i < 0$. Note f is bijective. □

Theorem

The set of all ordered pairs of natural numbers (n, m) is countable.

Proof.

Consider listing the pairs in order by their sum $s = n + m$, then by n. Every pair appears once in this series; the generating function is bijective. □

Uncountable Sets: Example

Theorem

The open interval $[0, 1) :\equiv \{r \in \mathbb{R} | 0 \leq r < 1\}$ *is uncountable.*

Proof by diagonalization: (Cantor, 1891).

- *Assume there is a series* $\{r_i\} = r_1, r_2, \cdots$ *containing all elements* $r \in [0, 1)$.
- *Consider listing the elements of* $\{r_i\}$ *in decimal notation (although any base will do) in order of increasing index: ...* *(continued on next slide)*

□

Uncountability of Reals, cont'd

(Cont.)

- A postulated enumeration of the reals:

$$
\begin{aligned}
r_1 &= 0.d_{1,1}d_{1,2}d_{1,3}d_{1,4}d_{1,5}d_{1,6}d_{1,7}d_{1,8}\ldots \\
r_2 &= 0.d_{2,1}d_{2,2}d_{2,3}d_{2,4}d_{2,5}d_{2,6}d_{2,7}d_{2,8}\ldots \\
r_3 &= 0.d_{3,1}d_{3,2}d_{3,3}d_{3,4}d_{3,5}d_{3,6}d_{3,7}d_{3,8}\ldots \\
r_4 &= 0.d_{4,1}d_{4,2}d_{4,3}d_{4,4}d_{4,5}d_{4,6}d_{4,7}d_{4,8}\ldots \\
\vdots & \qquad \vdots
\end{aligned}
$$

- Now, consider a real number generated by taking all digits $d_{i,i}$ that lie along the *diagonal* in this figure and replacing them with *different* digits.
- That real doesn't appear in the list!

□

Uncountability of Reals, fin.

(Fin.)

- E.g., a postulated enumeration of the reals:

$$
\begin{aligned}
r_1 &= 0.301948571\ldots\\
r_2 &= 0.103918481\ldots\\
r_3 &= 0.039194193\ldots\\
r_4 &= 0.918237461\ldots\\
\vdots &\qquad \vdots
\end{aligned}
$$

- OK, now let's add 1 to each of the diagonal digits$\pmod{10}$, that is changing 9's to 0.
- 0.4103... can't be on the list anywhere!

□

What are Strings, Really?

- This book says "finite sequences of the form $a_1, a_2, \cdots, a_n$ are called *strings*", but *infinite* strings are also used sometimes.
- Strings are often restricted to sequences composed of *symbols* drawn from a finite *alphabet*, and may be indexed from 0 or 1.
- Either way, the length of a (finite) string is its number of terms (or of distinct indexes).

Strings, more formally

- Let $\sum$ be a finite set of *symbols*, i.e. an *alphabet*.
- A *string* s over alphabet $\sum$ is any sequence $\{s_i\}$ of symbols, $s_i \in \sum$, indexed by $\mathbb{N}$ or $\mathbb{N} - \{0\}$.
- If $a, b, c, \cdots$ are symbols, the string $s = a, b, c, \cdots$ can also be written $abc \cdots$ (i.e., without commas).
- If s is a finite string and t is a string, the *concatenation* of s with t, written st, is the string consisting of the symbols in s, in sequence, followed by the symbols in t, in sequence.

More String Notation

- The length $|s|$ of a finite string s is its number of *positions* (i.e., its number of index values i).
- If s is a finite string and $n \in N$, s^n denotes the concatenation of n copies of s.
- ε denotes the empty string, the string of length 0.
- If $\sum$ is an alphabet and $n \in N$, then

$$\sum^{n} \equiv \{s \mid s \text{ is a string over } \sum \text{ of length } n\}, \text{ and}$$
$$\sum^{*} \equiv \{s \mid s \text{ is a finite string over } \sum\}.$$