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Information Processing Letters 92 (2004) 53–56

Information Processing Letters

www.elsevier.com/locate/ipl

Darts and hoopla board design

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Received 6 August 2003; received in revised form 2 September 2003

Available online 7 August 2004

Communicated by L.A. Hemaspaandra

Abstract

Dartboard design can be seen as an instance of the travelling salesman problem with maximum costs. This paper presents a simple yet optimal greedy algorithm to arrange numbers on both circular dartboards and linear hoopla boards. As a result, it identifies a class of polynomially solvable travelling salesman problems.

Keywords: Algorithms; Combinatorial problems; Travelling salesman

1. Board arrangements

Dartboard design [5,6] concerns the assignment of numbers to sectors of a circular board. For example, the traditional arrangement is shown in Fig. 1.

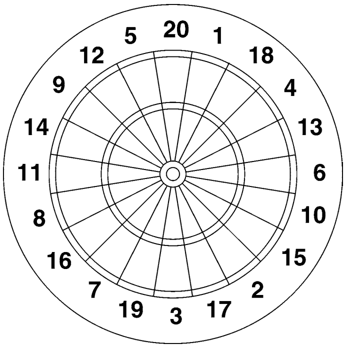

Fig. 1.

As thrown darts tend to land in either the aimed-at sector, or an adjacent one, a game of darts is arguably more exciting, the more adjacent pairs of numbers differ. Thus, given a bag of numbers $A = \{|a_1, a_2, \ldots, a_n|\}$ to arrange around the circumference of a dartboard, for a particular permutation $s = \alpha_1\alpha_2\ldots\alpha_n$ of A we define the function *risk* $s = \sum_{i=1}^{n} |\alpha_i - \alpha_{i-1}|^q$, where $\alpha_0 \equiv \alpha_n$ and q is a real constant such that $q \geqslant 1$. Note that bags are used because the numbers to be arranged may not be distinct, and as board rotations and reflections do not affect *risk*, it does not matter which sector is listed first, nor whether the numbers are listed in clockwise or counter-clockwise order. As $q \geqslant 1$, more weight is given to larger differences between adjacent numbers.

A frequently considered measure of good dartboard design is the function *risk*, and thus we define the dartboard design problem:

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0020-0190/$ – see front matter
doi:10.1016/j.ipl.2004.06.005

> Given a bag A of numbers and a real constant $q \geqslant 1$, it is required to produce a permutation of the numbers in A that maximises the value of the function *risk*.

Thus the dartboard design problem is an instance of a maximum-cost Travelling Salesman Problem (TSP).

Circular arrangements of numbers have been considered previously by several authors: Eiselt and Laporte [4] used a branch-and-bound algorithm [1] to find optimal arrangements of the standard dartboard numbers $\{1, 2, \ldots, 20\}$ for $q = 1, 2$, revealing that the traditional dartboard arrangement is good with respect to this measure, but not optimal. Chao and Liang [2] analysed circular permutations of n distinct numbers with $q = 1$, giving an exact characterization of both maximum and minimum arrangements with respect to the function *risk*. Subsequently, Cohen and Tonkes [3] analysed optimal arrangements for general bags (rather than sets) of numbers, using a string reversal algorithm.

This paper uses a different algorithm to generate optimal dartboard designs: a simple greedy algorithm. This results in an easy verification of the optimality of certain arrangements, and the algorithm can also be adapted to linear (hoopla) boards, yielding similar results. It follows that this particular TSP can be in solved in polynomial time.

2. A greedy algorithm

Let A be a bag of real numbers to be arranged around a dartboard. A greedy algorithm to perform the arranging can be described as follows:

> Initially, the minimum ($\sqcap A$) or maximum ($\sqcup A$) of the numbers is selected, to form a partial arc of the dartboard (a sequence of length 1). A complete board is then constructed by repeatedly adding a number to either end of the partial arc, until all numbers are used up. These numbers are chosen in a greedy fashion, so as to always make the biggest possible difference between the number selected and the number it is placed next to.

Note that each number placed (after the first) will either be the maximum of the remaining numbers, placed next to the smallest-numbered arc end, or the minimum, placed next to the largest-numbered arc end. The algorithm runs in $O(n \log n)$ time for a bag of n numbers: an initial sorting of the numbers takes $O(n \log n)$, and the greedy algorithm takes $O(n)$.

The proof of the greedy algorithm's correctness requires the following lemma:

Lemma. *Let $l_{\min}, l_{\max}, r_{\min}, r_{\max}, q$ be real numbers, with $q \geqslant 1$.*

If $l_{\min} \leqslant l_{\max}$ and $r_{\min} \leqslant r_{\max}$, then

$$|l_{\max} - r_{\min}|^q + |l_{\min} - r_{\max}|^q \geqslant |l_{\max} - r_{\max}|^q + |l_{\min} - r_{\min}|^q. \tag{1}$$

Proof. For $c \geqslant 0$ and $q \geqslant 1$, define the function $f(x) = (x + c)^q - x^q$, and note that f is non-decreasing and non-negative in the range $x \geqslant 0$. As the required inequality (1) is symmetrical, we can without loss of generality assume that $l_{\max} \geqslant r_{\max}$. There are then three cases:

Case $l_{\min} \leqslant r_{\min} \leqslant r_{\max} \leqslant l_{\max}$:

Let $a = r_{\min} - l_{\min}$, $c = r_{\max} - r_{\min}$ and $b = l_{\max} - r_{\max}$, so that $a, b, c \geqslant 0$. Then (1) $\equiv (b + c)^q + (a + c)^q \geqslant b^q + a^q \equiv f(b) + f(a) \geqslant 0$, which is true as f is non-negative for non-negative values.

Case $r_{\min} \leqslant l_{\min} \leqslant r_{\max} \leqslant l_{\max}$:

Let $a = l_{\min} - r_{\min}$, $b = r_{\max} - l_{\min}$ and $c = l_{\max} - r_{\max}$, so that $a, b, c \geqslant 0$. Then (1) $\equiv (a + b + c)^q + b^q \geqslant c^q + a^q \Leftarrow (a + c)^q \geqslant c^q + a^q \equiv f(a) \geqslant f(0)$, which is true as f is non-decreasing.

Case $r_{\min} \leqslant r_{\max} \leqslant l_{\min} \leqslant l_{\max}$:

Let $a = r_{\max} - r_{\min}$, $b = l_{\min} - r_{\max}$, $c = l_{\max} - l_{\min}$, so that $a, b, c \geqslant 0$. Then (1) $\equiv (a+b+c)^q + b^q \geqslant (b + c)^q + (a + b)^q \equiv f(a + b) \geqslant f(b)$, which is true as f is non-decreasing. $\square$

We may now prove that the greedy algorithm solves the dartboard design problem:

Proof. Given a bag of real numbers A to arrange on a dartboard, we first note that the choice of the minimum ($\sqcap A$) or maximum ($\sqcup A$) as the first number on the board is feasible, as this still leaves available all possible arrangements. Secondly, as the greedy algorithm progresses, the following invariant holds:

$$\forall \alpha \in s\colon\ (\alpha \leqslant \sqcap U) \vee (\alpha \geqslant \sqcup U) \tag{2}$$

where s is the sequence of numbers placed on the dartboard so far, and U is the bag of remaining unplaced numbers. The invariant holds initially because either s contains solely $\sqcap A$ and $U = A - \{|\sqcap A|\}$, or s contains solely $\sqcup A$ and $U = A - \{|\sqcup A|\}$. The invariant is maintained during the algorithm, as the maximum or minimum number in U is selected at each step and added to s.

Now consider a partial dartboard arrangement, during the progress of the algorithm, after zero or more greedy choices have been made. It suffices to show that if the arrangement so far can be completed optimally with respect to the function *risk*, then performing the greedy choice retains this property.

Let the greedy step choose to add the number a_i (of the remaining numbers in U) to the partial dartboard arrangement so far, s. As the situation is symmetrical, without loss of generality suppose that $a_i = \sqcup U$ and that the greedy step appends a_i to the end of the sequence s, as sa_i. Taking a_j as the last number in s, a_j must be less than or equal to the first number of s, because otherwise the greedy choice would have chosen to put a_i next to the first number in s. In addition, for all $a_k \in U$, $a_j \leqslant a_k \leqslant a_i$, because $a_i = \sqcup U$ and the above invariant holds.

Now consider any optimal completion of s (not necessarily involving greedy choices). If a_i is not placed next to a_j in this arrangement, this completion must be of the form shown in Fig. 2.

It may be that a_l is the first number in the sequence s, or $a_l \in U$, but in either case, as noted above, $a_j \leqslant a_l$. This completed dartboard is now "cut" in two places: between a_j and a_k, and between a_l and a_i. The arc $a_k t a_i$ is then reversed within the dartboard arrangement, thus changing adjacent number differences only at a_j, a_k and a_l, a_i. Within

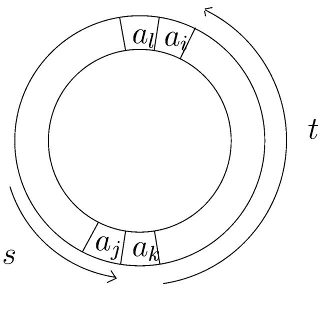

Fig. 2.

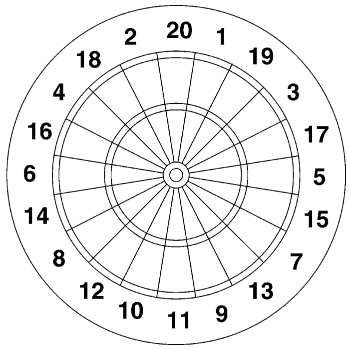

Fig. 3.

this new arrangement, as $a_j \leqslant a_l$ and $a_k \leqslant a_i$, use of the above lemma assures that this is no worse with respect to the function *risk*. Therefore the greedy choice can also lead to an optimal completion. □

Therefore, if the numbers $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ are to be arranged around a dartboard, then an arrangement of the form $\ldots x_5 x_{n-3} x_3 x_{n-1} x_1 x_n x_2 x_{n-2} x_4 x_{n-4} \ldots$ is clearly optimal, as it can be produced by the greedy algorithm. Here is this arrangement for the numbers $\{|1 \ldots 20|\}$ (see Fig. 3).

3. Hoopla variations

Instead of using a circular board, one might try arranging numbers in a linear fashion. Such a board can be used for the common fairground game hoopla, which involves throwing hoops over target pegs, which each have different scores. The same principles of risk and excitement can be said to apply similarly to hoopla, and therefore a hoopla proprietor might be interested in maximising the function $risk_2\ s = \sum_{i=2}^{n} |\alpha_i - \alpha_{i-1}|^q$, for a particular permutation $s = \alpha_1 \alpha_2 \ldots \alpha_n$ of a bag of numbers.

Exactly the same greedy algorithm works for this variation: first selecting the maximum or minimum number, and then at each step, adding a number to the end of the board segment so far to form the largest possible difference. This time, the sequence is not regarded as circular. The same argument is used to prove the correctness of this algorithm: the "arc reversal" step on the optimal completion (completing to a linear dartboard) is performed in exactly the same way as if it were representing a circular dartboard.

The hoopla proprietor gets a similar result to the dartboard arranger: for example, the arrangement $\ldots x_5 x_{n-3} x_3 x_{n-1} x_1 x_n x_2 x_{n-2} x_4 x_{n-4} \ldots$ is one possible optimal solution produced by the greedy algorithm from the numbers $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$, with the exact ends of the sequence being dictated by the last step of the greedy algorithm.

A more cunning hoopla proprietor, wishing to lower competitors' scores to avoid giving away prizes, might think that a better model for the hoopla board would be $0s0$ (where s is a sequence giving an permutation of the original bag of numbers), as any hoops thrown past the end of the board score 0 anyway.

Note that such a linear board $0s0$ has the same risk as the circular arrangement described by $s0$. Thus, if the numbers to be arranged on the linear board are all non-negative, the greedy algorithm for the circular board can be used, if it starts from the partial arc of size 1 containing 0.

Thus the arrangements produced by this second variation are of a different form, with the extreme numbers at the ends, not in the middle. For example, $0 x_n x_1 x_{n-2} \ldots x_{n-3} x_2 x_{n-1} 0$ is one possible optimal arrangement produced by the greedy algorithm from the numbers $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$.

4. Conclusion

Whilst knowledge of optimal dartboard arrangements with respect to this measure has existed for some time [2,3], this presentation makes use of a new algorithm, which is a greedy algorithm. This results in a short proof and simple identification of optimal arrangements.

It is already known that the dartboards problem can be formulated as a maximum-cost travelling salesman problem, from Eiselt and Laporte [4]. Thus this paper identifies a class of travelling salesman problems solvable in polynomial time.

Acknowledgements

Thanks to the anonymous referees for their helpful comments.

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