Chapter 8 The Discrete Fourier Transform

- Introduction
- Definition of the Discrete Fourier Transform
- Properties of the DFT
- Linear Convolution Using the DFT
- DCT
- Concluding Remarks
1. Introduction

- Fourier Transform Pair
  \[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \exp(-jn\omega) \]
  \[ x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{jn\omega} d\omega \]

- Required Transform
  - Finite Signal Length
  - Discrete Frequency Samples

- DFT Applications
  - Signal Analysis
  - Computing benefits from the convolution
2. Discrete Fourier Transform

- Considerations
  - The Discrete-time Fourier transform is defined for sequences with finite or infinite length.
  - The DFT is defined only for sequences with finite length.

- Formulation-- Forward Transform
  - Consider the sequence of length $N$ $x[n]$ for $n=0,1,2,...,N-1$
  - The discrete-time Fourier transform is
    \[
    X(\omega) = \sum_{k=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{k=0}^{N-1} x[n] e^{-j\omega n}
    \]
  - It is periodic with period $2\pi$. 
2. Discrete Fourier Transform (c.1)

- **Formulation-- Forward Transform (c.1)**
  - The usually considered frequency interval is \((-\pi, \pi]\). There are infinitely many points in the interval. If \(x[n]\) has \(N\) points, we compute \(N\) equally spaced \(\omega\) in \((-\pi, \pi]\). These \(N\) points are
    \[
    \tilde{X}(k) = X(k \frac{2\pi}{N}) = \sum_{n=0}^{N-1} x[n] e^{-jn2\pi/N} \quad \omega = k \frac{2\pi}{N}, \quad k = 0, 1, 2, \ldots, N - 1
    \]
  - So we define
    \[
    x[n] = \frac{1}{2\pi} \int_{-\omega}^{\omega} X(\omega) e^{jn\omega} d\omega \approx \frac{1}{2\pi} \sum_{k=0}^{N-1} X(\omega_k) e^{-jk2\pi/N} \cdot \left( \frac{2\pi}{N} \right)
    \]
    where \(k\) and \(n\) are in the interval from 0 to \(N-1\).

- **Formulation-- Inverse Transform**
  - Compute \(x[n]\) from \(N\) frequency components
    
    So
    \[
    x[n] \approx \frac{1}{N} \sum_{m=0}^{N-1} X(k) e^{-jkn2\pi/N}
    \]
2. Discrete Fourier Transform (c.2)

- The Discrete Fourier Transform

\[ X(k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x[n] W^{kn} \]

\[ x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X(k) e^{jkn 2\pi / N} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W^{-nk} \]

where \( W = e^{-j2\pi / N} \); \( k, n = 0, 1, 2, \ldots, N - 1 \)

- Notes

- **Periodic property of X[k]**
  - \( W^{knN} = 1; X[k] = X[k+N] = X[k+2N] = \ldots \)

- **DFT** compute the samples of the discrete-time Fourier transform of \( x[n] \).

- **The Inverse yields a periodic \( x[n] \) with period \( N \)**

\[ \overline{X}[k] = \begin{cases} X[k] & \text{for } 0 \leq k \leq N - 1 \\ \text{periodic extension of } X[k] & \end{cases} \]

- **The relation with the Discrete Fourier series**
  - \( X[k] = Nc_k \)
2. Properties of the DFT

Let \( x'[n] \) is the periodic extension of the \( x[n] \)

- **Linearity Properties**
  \( \mathcal{D}[a_1x_1[n] + a_2x_2[n]] = a_1 \mathcal{D}[x_1[n]] + a_2 \mathcal{D}[x_2[n]] \)

- **Symmetric Properties**
  - If \( x'[n] \) is real then \( X[k] \) is conjugate symmetric, or \( X[-k] = X^*[k] \).
  - If \( x'[n] \) is real and even, then \( X[k] \) is real and even.
  - If \( x'[n] \) is real and odd, then \( X[k] \) is imaginary and odd.

- **Duality Properties**
  - Let \( X[k] \) and \( x'[n] \) be the DFT pair. That is
    \( X[k] = \mathcal{D}[x'[n]] \) and \( x'[n] = \mathcal{D}^{-1}[X[k]] \).
  
    then
    \( x'[n] = \{\mathcal{D}[X[k]/N]\}^* \)
  - The DFT can be used to compute the inverse DFT.

- **Time-Shifting & Frequency-Shifting Properties**
  - \( \mathcal{D}[x'[n-n_0]] = e^{-jkn_0} X[k] \) & \( \mathcal{D}[x'[n]e^{jnk_0}] = X[k-k_0] \)
2. Properties of the DFT-- Periodic Convolution

- **Topic**
  - Compute convolution of two finite sequence using the DFT

- **Consideration**
  - **Convolution**
    - Convolution of two sequences \( h[n] \) & \( u[n] \) for \( n=0, 1, 2, ..., N-1 \).
      \[
      y[n] = \sum_{i=0}^{n} h[n-i]u[i] = \sum_{i=0}^{n} h[i]u[n-i]
      \]
    - The nonzero elements are at most in the interval \([0, 2N-2]\).
  - **Periodic or Cyclic Convolution**
    - If \( u_1[n] \) and \( u_2[n] \) are two periodic sequence with period \( N \). The cyclic convolution of the two sequence is defined as
      \[
      y[k] = \sum_{i=0}^{N-1} u_1[k-i]u_2[i] = \sum_{i=0}^{N-1} u_1[i]u_2[k-i]
      \]
      where \( n=0, 1, 2, ..., N-1 \).
  - **Cyclic Convolution & DFT**
    - The DFT of \( y[n] \) is equal to the multiplication of the DFT of \( u_1[n] \) and \( u_2[n] \).
      \[
      Y[k] = U_1[k]U_2[k]
      \]
2. Properties of the DFT -- Periodic Convolution (c.1)
Figure 8.3 Procedure for forming the periodic convolution of two periodic sequences.
2. Properties of the DFT-- Periodic Convolution (c.2)

Definition

\[ X(k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x[n]W^{kn} \]

\[ x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X(k)e^{j2\pi n kn} = \frac{1}{N} \sum_{k=0}^{N-1} X(k)W^{-nk} \]

where \( W = e^{-j2\pi/N} \); \( k, n = 0, 1, 2, \ldots, N-1 \)

<table>
<thead>
<tr>
<th>Finite-Length Sequence (Length N)</th>
<th>N-Point DFT (Length N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( x[n] )</td>
<td>( X[k] )</td>
</tr>
<tr>
<td>2. ( x_1[n], x_2[n] )</td>
<td>( X_1[k], X_2[k] )</td>
</tr>
<tr>
<td>3. ( ax_1[n] + bx_2[n] )</td>
<td>( aX_1[k] + bX_2[k] )</td>
</tr>
<tr>
<td>4. ( X[n] )</td>
<td>( Nx[(-k)n] )</td>
</tr>
<tr>
<td>5. ( x[((n - m)n)] )</td>
<td>( W_n^m X[k] )</td>
</tr>
<tr>
<td>6. ( W_n^m x[n] )</td>
<td>( X[((k - \ell)n)] )</td>
</tr>
<tr>
<td>7. ( \sum_{n=0}^{N-1} x_1(m)x_2[((n - m)n)] )</td>
<td>( X_1[k]X_2[k] )</td>
</tr>
<tr>
<td>8. ( x_1[n]x_2[n] )</td>
<td>( \frac{1}{N} \sum_{\ell=0}^{N-1} X_1(\ell)X_2[((k - \ell)n)] )</td>
</tr>
<tr>
<td>9. ( x^*[n] )</td>
<td>( X^*[((-k)n)] )</td>
</tr>
<tr>
<td>10. ( x^*[((-n)n)] )</td>
<td>( X^*[k] )</td>
</tr>
<tr>
<td>11. ( \mathbb{R}{x[n]} )</td>
<td>( X_{ep}[k] = \frac{1}{2}(X[((k)n)] + X^*[((-k)n)]) )</td>
</tr>
<tr>
<td>12. ( j\mathbb{M}{x[n]} )</td>
<td>( X_{ep}[k] = \frac{1}{2}(X[((k)n)] - X^*[((-k)n)]) )</td>
</tr>
<tr>
<td>13. ( x_{ep}[n] = \frac{1}{2}(x[n] + x^*[((-n)n)]) )</td>
<td>( \mathbb{R}{X[k]} )</td>
</tr>
<tr>
<td>14. ( x_{ep}[n] = \frac{1}{2}(x[n] - x^*[((-n)n)]) )</td>
<td>( j\mathbb{M}{X[k]} )</td>
</tr>
</tbody>
</table>

Properties 15–17 apply only when \( x[n] \) is real.

15. Symmetry properties

\[
\begin{align*}
X[k] &= X^*[((-k)n)] \\
\mathbb{R}{X[k]} &= \mathbb{R}{X[((k)n)]} \\
\mathbb{M}{X[k]} &= -j\mathbb{M}{X[((k)n)]} \\
\end{align*}
\]

16. \( x_{ep}[n] = \frac{1}{2}(x[n] + x[((-n)n)]) \) \( \mathbb{R}{X[k]} \)

17. \( x_{ep}[n] = \frac{1}{2}(x[n] - x[((-n)n)]) \) \( j\mathbb{M}{X[k]} \)
3. Implementing LTI Systems Using the DFT

- **Linear Convolution**
  - Finite length $h[n]$ and indefinite-length $x[n]$

- **Method 1**
  - Nonoverlapping-and-add

\[
x[n] = \sum_{r=0}^{\infty} x_r[n - rL]
\]

where

\[
x_r[n] = \begin{cases} x[n + rL], & 0 \leq n \leq L - 1 \\ 0, & \text{otherwise} \end{cases}
\]
3. Implementing LTI Systems Using the DFT (c.1)

- Method 1 (c.1)
  \[ y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} y_r[n - rL] \]
  \[ y_r[n] = x_r[n] * h[n] \]

- Method 2
  - Overlapping-and-skipping
    \[ x_r[n] = x[n + r(L-P+1) - P+1], \quad 0 \leq n \leq L-1 \]
  - The output
    \[ y[n] = \sum_{r=0}^{\infty} y_r[n - r(L-P+1) + P-1] \]
  where
    \[ y_r[n] = \begin{cases} y_{rp}[n], & P-1 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases} \]
The Discrete Cosine Transform (DCT)

- General Transform

\[ A[k] = \sum_{n=0}^{N-1} x[n]\phi_k^*[n], \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} A[k]\phi_k[n]. \]

\[ \phi_k[n], \text{ referred to as the basis sequences,} \]

\[ \frac{1}{N} \sum_{n=0}^{N-1} \phi_k[n]\phi_m^*[n] = \begin{cases} 1, & m = k, \\ 0, & m \neq k. \end{cases} \]

- Four types of DCT is defined for different periodicity

Type I

Type II

Type III

Type IV
The Discrete Cosine Transform (DCT)

- **Type I DCT**
  - Extend $x(n)$ by
    
    $$\tilde{x}_1[n] = x_\alpha[((n))_{2N-2}] + x_\alpha[(-n)_{2N-2}],$$
    
    $$: x_\alpha[n] = \alpha[n]x[n] \quad \alpha[n] = \begin{cases} 
    \frac{1}{2}, & n = 0 \text{ and } N - 1, \\
    1, & 1 \leq n \leq N - 2. 
    \end{cases}$$

  - We have
    
    $$X^{c1}[k] = 2 \sum_{n=0}^{N-1} \alpha[n]x[n] \cos \left( \frac{\pi kn}{N-1} \right), \quad 0 \leq k \leq N - 1,$$
    
    $$x[n] = \frac{1}{N-1} \sum_{k=0}^{N-1} \alpha[k]X^{c1}[k] \cos \left( \frac{\pi kn}{N-1} \right), \quad 0 \leq n \leq N - 1,$$
The Discrete Cosine Transform (DCT)

- Type II DCT
  - Extend \( x(n) \) by
    \[
    \tilde{x}_2[n] = x[((n))_{2N}] + x[(-n - 1)]_{2N},
    \]
  - We have
    \[
    X^c2[k] = 2 \sum_{n=0}^{N-1} x[n] \cos \left( \frac{\pi k (2n + 1)}{2N} \right), \quad 0 \leq k \leq N - 1,
    \]
    \[
    x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \beta[k] X^c2[k] \cos \left( \frac{\pi k (2n + 1)}{2N} \right), \quad 0 \leq n \leq N - 1,
    \]
    \[
    \beta[k] = \begin{cases} 
    \frac{1}{2}, & k = 0 \\
    1, & 1 \leq k \leq N - 1.
    \end{cases}
    \]
The Discrete Cosine Transform (DCT)

- Other Variant with different normalization factor

\[
\tilde{X}^{c2}[k] = \sqrt{\frac{2}{N}} \tilde{\beta}[k] \sum_{n=0}^{N-1} x[n] \cos \left( \frac{\pi k (2n + 1)}{2N} \right), \quad 0 \leq k \leq N - 1,
\]

\[
x[n] = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} \tilde{\beta}[k] \tilde{X}^{c2}[k] \cos \left( \frac{\pi k (2n + 1)}{2N} \right), \quad 0 \leq n \leq N - 1,
\]

\[
\tilde{\beta}[k] = \begin{cases} 
\frac{1}{\sqrt{2}}, & k = 0, \\
1, & k = 1, 2, \ldots, N - 1.
\end{cases}
\]
The Discrete Cosine Transform (DCT)

- Energy Compaction

\[ x[n] = a^n \cos(\omega_0 n + \phi), \quad n = 0, 1, \ldots, N - 1. \]

\[ a = 0.9, \omega_0 = 0.1\pi, \phi = 0, \text{ and } N = 32. \]
Figure 8.28 (a) Real part of $N$-point DFT; (b) Imaginary part of $N$-point DFT; (c) $N$-point DCT-2 of the test signal plotted in Figure 8.27.
DCT

- Error Comparison

\[ x_m^{\text{det}}[n] = \frac{1}{N} \sum_{k=0}^{N-1-m} \beta[k] X^c[k] \cos \left( \frac{\pi k (2n + 1)}{2N} \right), \quad 0 \leq n \leq N - 1. \]

\[ E^{\text{dft}}[m] = \frac{1}{N} \sum_{n=0}^{N-1} |x[n] - x_m^{\text{dft}}[n]|^2 \]

\[ E^{\text{det}}[m] = \frac{1}{N} \sum_{n=0}^{N-1} |x[n] - x_m^{\text{det}}[n]|^2 \]

![Graph showing comparison between DCT and DFT truncation errors](image)
Homeworks

- Deadline: May 11, Class time
  - 8.31, 8.33, 8.40, 8.41, 8.42