Chapter 7 Filter Design Techniques

- Introduction
- Design of FIR Filters by Windowing
- Examples of FIR Filter Design by the Kaiser Window Method
- Design of Discrete-Time IIR Filters from Continuous-Time Filters
- Frequency Transformations of Lowpass IIR Filters
- Appendix Continuous Filters
1. Introduction

- Filters
  - Frequency-selective Filter

- Three Stages
  - Specifications
  - Approximation of the Spec.
  - Realization

- Ex.
  - Spec. $\delta_1 = 0.01 \ (20 \log_{10} (1 + \delta_1) = 0.086 \text{ dB}),$
    $\delta_2 = 0.001 \ (20 \log_{10} \delta_2 = -60 \text{ dB}),$
    $\Omega_p = 2\pi(2000),$
    $\Omega_s = 2\pi(3000).$
2. Design of FIR Filters by Windowing

- **Procedure**
  - Desired Frequency Response
    \[ H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n} \]
    \[ h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega} d\omega \]
  - Corresponding Impulse Response
    \[ h[n] = \begin{cases} h_d[n], & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \]
  - Length is Infinity
  - Truncation
2. Design of FIR Filters by Windowing (c.1)

- **Analysis**
  - **Windowing in Time**
    
    \[
    h[n] = h_d[n]w[n]
    \]
    
    where
    
    \[
    w[n] = \begin{cases} 
    1, & 0 \leq n \leq M \\
    0, & \text{otherwise}
    \end{cases}
    \]

  - **Convolution in Frequency**
    
    \[
    H(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)}) \, d\theta
    \]

- **Points**
  - Width of the Mainlobe
  - Attenuation of the Sidelobe

- **Rect. Window**
  
  \[
  W(e^{j\omega}) = \sum_{n=0}^{M} e^{-jn\omega} = \frac{1 - e^{-j(M+1)\omega}}{1 - e^{-j\omega}} = e^{-jM\omega/2} \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}
  \]

- **Freq. Resp.**

- **Convolution**

- **Results**
Properties of Commonly Used Windows

**Rectangular**

\[ w[n] = \begin{cases} 1, & 0 \leq n \leq M, \\ 0, & \text{otherwise} \end{cases} \]

**Bartlett (triangular)**

\[ w[n] = \begin{cases} 2n/M, & 0 \leq n \leq M/2, \\ 2 - 2n/M, & M/2 < n \leq M, \\ 0, & \text{otherwise} \end{cases} \]

**Hanning**

\[ w[n] = \begin{cases} 0.5 - 0.5 \cos(2\pi n/M), & 0 \leq n \leq M, \\ 0, & \text{otherwise} \end{cases} \]

**Hamming**

\[ w[n] = \begin{cases} 0.54 - 0.46 \cos(2\pi n/M), & 0 \leq n \leq M, \\ 0, & \text{otherwise} \end{cases} \]

**Blackman**

\[ w[n] = \begin{cases} 0.42 - 0.5 \cos(2\pi n/M) + 0.08 \cos(4\pi n/M), & 0 \leq n \leq M, \\ 0, & \text{otherwise} \end{cases} \]
2. Design of FIR Filters by Windowing (c.3)

- Properties of Commonly Used Windows (c.1)

<table>
<thead>
<tr>
<th>Window Type</th>
<th>Peak Sidelobe Amplitude (Relative)</th>
<th>Approximate Width of Mainlobe</th>
<th>Peak Approximation Error $20 \log_{10} \delta$ (dB)</th>
<th>Equivalent Kaiser Window $\beta$</th>
<th>Transition Width of Equivalent Kaiser Window</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>$-13$</td>
<td>$4\pi/(M + 1)$</td>
<td>$-21$</td>
<td>$0$</td>
<td>$1.81\pi/M$</td>
</tr>
<tr>
<td>Bartlett</td>
<td>$-25$</td>
<td>$8\pi/M$</td>
<td>$-25$</td>
<td>$1.33$</td>
<td>$2.37\pi/M$</td>
</tr>
<tr>
<td>Hanning</td>
<td>$-31$</td>
<td>$8\pi/M$</td>
<td>$-44$</td>
<td>$3.86$</td>
<td>$5.01\pi/M$</td>
</tr>
<tr>
<td>Hamming</td>
<td>$-41$</td>
<td>$8\pi/M$</td>
<td>$-53$</td>
<td>$4.86$</td>
<td>$6.27\pi/M$</td>
</tr>
<tr>
<td>Blackman</td>
<td>$-57$</td>
<td>$12\pi/M$</td>
<td>$-74$</td>
<td>$7.04$</td>
<td>$9.19\pi/M$</td>
</tr>
</tbody>
</table>
2. Design of FIR Filters by Windowing (c.4)

- **Generalized Linear Phase**
  - Symmetric of the Windowing

\[ w(n) = \begin{cases} w(M-n), & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \]
2. Design of FIR Filters by Windowing (c.5)

- The Kaiser Window Filter
  - Zeroth-order Bessel Function of the First Kind
    \[ w(n) = \left\{ \begin{array}{ll}
    I_0 \left[ \beta (1 - [(n - \alpha)/\alpha]^2)^{1/2} \right] & , \quad 0 \leq n \leq M \\
    I_0(\beta) & , \quad \alpha = M/2 \\
    0, & \text{otherwise}
    \end{array} \right. \]

- Two Parameters
  - \( \alpha \) and \( \beta \)
Defined by the mathematician Daniel Bernoulli and generalized by Friedrich Bessel, are canonical solutions $y(x)$ of Bessel's differential equation.

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

$$J_{\alpha}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$
2. Design of FIR Filters by Windowing

(c.6)

- The Kaiser Window Filter -- Procedure
  - Evaluate first the two values
    \[ \Delta \omega = \omega_s - \omega_p \]
    \[ A = -20 \log_{10} \delta \]
  - Value of \( \beta \) (empirically) and \( M \)
    \[ \beta = \begin{cases} 
    0.1102(A - 8.7), & A > 50, \\
    0.5842(A - 21)^{0.4} + 0.07886(A - 21), & 21 \leq A \leq 50, \\
    0.0, & A < 21.
    \end{cases} \]
    \[ M = \frac{A - \delta}{2.285 \Delta \omega} \]
  - Example
    \[ \omega_p = 0.4\pi \quad \omega_s = 0.6\pi \quad \delta_1 = 0.01 \quad \delta_2 = 0.001 \]
    \[ \Delta \omega = 0.2\pi \quad \delta = 0.001 \quad \beta = 5.653 \quad M = 37 \]

\[ h[n] = \begin{cases} 
\frac{\sin \omega_c (n - \alpha)}{\pi (n - \alpha)} \frac{I_0[\beta(1 - [(n - \alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)} & 0 \leq n \leq M \\
0, & \text{otherwise}
\end{cases} \]
3. Examples of FIR Filter Design by the Kaiser Window Method

- Highpass Filter
  - Freq. Resp.
    \[ H_{hp}(e^{j\omega}) = \begin{cases} 
    0, & 0 \leq |\omega| < \omega_c \\
    e^{-j\omega M/2}, & \omega_c \leq |\omega| < \pi 
  \end{cases} \]

  \[ H_{hp}(e^{j\omega}) = e^{-j\omega M/2} - H_{lp}(e^{j\omega}) \]

  - Impulse Response
    - Impulse Response of the High pass Filters
      \[ h_{hp}[n] = \frac{\sin \pi (n - M/2)}{\pi (n - M/2)} - \frac{\sin \omega_c (n - M/2)}{\pi (n - M/2)}, \quad -\infty < n < \infty \]

- Generalized Multiband Filters
  \[ h_{mb}[n] = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin \omega_k (n - M/2)}{\pi (n - M/2)}, \quad -\infty < n < \infty \]
4. Design of Discrete-Time IIR Filters from Continuous-Time Filters

- **Motivation for Digital Filter from Analog Filter**
  - Analog IIR filter design is highly advanced.
  - Have relatively simple closed form design
  - The methods do not lead to a simple closed form design formulas for discrete-time IIR case.

- **Analog Filters**
  - An analog filter may be described as
    \[ H_a(s) = \frac{B(s)}{A(s)} = \sum_{k=0}^{M} B_k s^k \]
    \[ = \sum_{k=0}^{N} A_k s^k \]
  - Analog filter is stable if all its poles lie in the left-half of s-plane.
  - The jΩ axis in the s-plane \(\Rightarrow\) the unit circle in the z-plane.
  - The left-half plane of the s-plane \(\Rightarrow\) Inside the unit circle.
4. Design of Discrete-Time IIR Filters from Continuous-Time Filters-- Filter Design by Impulse Invariance

- **Concepts**
  - The sampling of Impulse response
    - $h[n] = T h_c(nT)$
  - **Freq. Relation**
    - $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c(j \frac{\omega}{T} + j \frac{2\pi}{T} k)$
    - $H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$
    - $h_c(t) = \begin{cases} 
    \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0 \\
    0, & t < 0 
    \end{cases}$
  - **Ex.**
    - $h[n] = T h_c(nT) = \sum_{k=1}^{N} TA e^{s_k n T_d} u[n] = \sum_{k=1}^{N} TA(e^{s_k T})^n u[n]$
    - $H(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - e^{s_k T} z^{-1}}$

- The mapping is not one-to-one.
- The selection of $T$ to minimize the aliasing.
- Appropriate for low-pass and band-pass filters.
4. Design of Discrete-Time IIR Filters from Continuous-Time Filters—Bilinear Transformation

**Concepts**

**Formula**

\[
\begin{align*}
    s &= \frac{2}{T} \left(1 - z^{-1}\right) \\
    z &= \frac{1 + (T/2)\sigma + j\Omega(T/2)}{1 - (T/2)\sigma - j\Omega(T/2)} \\
    H(z) &= H_c \left(\frac{2}{T} \left(1 - z^{-1}\right)\right) \\
    H(z) &= H_c \left(\frac{2}{T} \left(1 - z^{-1}\right)\right) \\
    s &= \sigma + j\Omega = \frac{2}{T} \left[\frac{2e^{-j\omega/2} (j\sin\omega/2)}{2e^{-j\omega/2} (\cos\omega/2)}\right] = \frac{2j}{T} \tan(\omega/2) \\
    \Omega &= \frac{2}{T} \left[\frac{2e^{-j\omega/2} (j\sin\omega/2)}{2e^{-j\omega/2} (\cos\omega/2)}\right] = \frac{2}{T} \tan(\omega/2) \\
    \omega &= 2 \arctan(\Omega T/2)
\end{align*}
\]

**Frequency Relation**

\[
\Omega = \frac{2}{T} \left[\frac{2e^{-j\omega/2} (j\sin\omega/2)}{2e^{-j\omega/2} (\cos\omega/2)}\right] = \frac{2}{T} \tan(\omega/2)
\]

**Frequency mapping**

\[
\omega = 2 \arctan(\Omega T/2)
\]
5. Frequency Transformations of Lowpass IIR Filters

**Transform**
- Replacing the variable $z^{-1}$ by a rational function
- $z^{-1} = g(z^{-1})$

**Conditions**
- Unit circle to unit circle.
- $e^{-j\omega} = g(e^{-j\omega}) = g(\omega)$
- $|g(\omega)|e^{j \text{arg}g(\omega)}$
- Inside unit circle to Inside unit circle

**Form**
- $g(z^{-1}) = \pm \prod_{k=1}^{n} \frac{Z^{-1} - \alpha_k}{1 - \alpha_k Z^{-1}}$
- $|\alpha_k| < 1$

Transform from lowpass digital filter prototype cutoff frequency $\theta$.

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Transformation</th>
<th>Associated Design Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lowpass</td>
<td>$Z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$</td>
<td>$\alpha = \frac{\sin(\frac{\theta - \omega_p}{2})}{\sin(\frac{\theta + \omega_p}{2})}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_p = \text{desired cutoff frequency}$</td>
</tr>
<tr>
<td>Highpass</td>
<td>$Z^{-1} = \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$</td>
<td>$\alpha = -\frac{\cos(\frac{\theta + \omega_p}{2})}{\cos(\frac{\theta - \omega_p}{2})}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_p = \text{desired cutoff frequency}$</td>
</tr>
<tr>
<td>Bandpass</td>
<td>$Z^{-1} = \frac{z^{-2} - \frac{2ak}{k+1}z^{-1} + \frac{k-1}{k+1}}{k-1} \div \frac{z^{-2} - \frac{2ak}{k+1}z^{-1} + \frac{k-1}{k+1}}{k+1}$</td>
<td>$\alpha = \frac{\cos(\frac{\omega_p + \omega_{p1}}{2})}{\cos(\frac{\omega_p - \omega_{p1}}{2})}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k = \cos(\frac{\omega_p - \omega_{p1}}{2}) \tan(\frac{\theta}{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_{p1} = \text{desired lower cutoff frequency}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_{p2} = \text{desired upper cutoff frequency}$</td>
</tr>
<tr>
<td>Bandstop</td>
<td>$Z^{-1} = \frac{z^{-2} - \frac{2a}{1+k}z^{-1} + \frac{1-k}{1+k}}{1-k} \div \frac{z^{-2} - \frac{2a}{1+k}z^{-1} + \frac{1-k}{1+k}}{1+k}$</td>
<td>$\alpha = \frac{\cos(\frac{\omega_p + \omega_{p1}}{2})}{\cos(\frac{\omega_p - \omega_{p1}}{2})}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k = \tan(\frac{\omega_p - \omega_{p1}}{2}) \tan(\frac{\theta}{2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_{p1} = \text{desired lower cutoff frequency}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_{p2} = \text{desired upper cutoff frequency}$</td>
</tr>
</tbody>
</table>
6. Concluding Remarks

- Introduction
- Design of FIR Filters by Windowing
- Examples of FIR Filter Design by the Kaiser Window Method
- Design of Discrete-Time IIR Filters from Continuous-Time Filters
- Frequency Transformations of Lowpass IIR Filters
Continuous-Time Filters

C.M. Liu
Perceptual Signal Processing Lab
College of Computer Science
National Chiao-Tung University

http://www.csie.nctu.edu.tw/~cmliu/Courses/DSP/

Office: EC538
(03)5731877
cmliu@cs.nctu.edu.tw
Contents

- Butterworth Lowpass Filters
- Chebyshev Filters
- Elliptic Filters
Butterworth Lowpass Filters

- **Maximally Flat Magnitude Filter**
  - Magnitude response is maximally flat as mathematically possible in the passband.
  - The Butterworth type filter was first described by the British engineer Stephen Butterworth in his paper "On the Theory of Filter Amplifiers".

\[
|H_c(j\Omega)|^2 = \frac{1}{1 + (j\Omega/j\Omega_c)^{2N}}.
\]
Butterworth Lowpass Filters

Let

\[ j\Omega = s \]

\[ H_c(s)H_c(-s) = \frac{1}{1+(s/j\Omega_c)^{2N}}. \]

The roots of the denominator polynomial

\[ 1 + (s/j\Omega_c)^{2N} = 0 \]

\[ s_k = (-1)^{1/2N}(j\Omega_c) = \Omega_ce^{(j\pi/2N)(2k+N-1)}, \quad k = 0, 1, \ldots, 2N - 1. \]
Chebyshev Filters

- A steeper roll-off and more passband ripple (type I) or stopband ripple (type II) than Butterworth filters.
- In honor of Pafnuty Chebyshev because their mathematical characteristics are derived from Chebyshev polynomials.

\[ |H_c(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 V_N^2(\Omega/\Omega_c)}, \]

where \( V_N(x) \) is the \( N \)th-order Chebyshev polynomial defined as

\[ V_N(x) = \cos(N \cos^{-1} x). \]

For example, for \( N = 0, V_0(x) = 1 \); for \( N = 1, V_1(x) = \cos(\cos^{-1} x) = x \); for \( N = 2, V_2(x) = \cos(2 \cos^{-1} x) = 2x^2 \) \( - 1 \); and so on.
Chebyshev Filters

\[ V_N^2(x) \text{ varies between zero and unity for } 0 < x < 1. \]
\[ V_{N+1}(x) = 2x V_N(x) - V_{N-1}(x). \]
\[ x > 1, \cos^{-1} x \text{ is imaginary, so } V_N(x) \text{ behaves as a hyperbolic cosine consequently increases monotonically}. \]
\[ a = \frac{1}{2}(\alpha^{1/N} - \alpha^{-1/N}) \quad \alpha = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}}. \]

The length of the major axis is \( 2b\Omega_c \).

\[ b = \frac{1}{2}(\alpha^{1/N} + \alpha^{-1/N}). \]
Chebyshev Filters

- Type II is related to Type I through a transformation.

\[ |H_c(j\Omega)|^2 = \frac{1}{1 + [\epsilon^2 V_N^2(\Omega_c/\Omega)]^{-1}}. \]
Elliptic Filters

- **Ripple in both passband and stopband**

\[ G_n(\omega) = \frac{1}{\sqrt{1 + \varepsilon^2 R_n^2(\xi, \omega/\omega_0)}} \]

This type of approximation is the best can be achieved for a given order. \( R_n \) is the \( n \)th-order elliptic rational function (sometimes known as a Chebyshev rational function) and

- \( \omega_0 \) is the cutoff frequency
- \( \varepsilon \) is the ripple factor
- \( \xi \) is the selectivity factor
Homeworks

- 7.6; 7.15; 7.16; 7.27; 7.31;