The z-Transform

Introduction

z-Transform

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Inverse z-Transform

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Solving the Difference Equations

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Summary
1. Introduction

Role in Discrete-Time Systems

- z-Transform is the discrete-time counterpart of the Laplace transform.

Response of Discrete-Time Systems

- If a system is lumped, it can be described as a difference equation
  \[ 2y[k] + 3y[k-1] + y[k-2] = u[k] + u[k-1] - u[k-2] \text{ for } k = 0, 1, 2 \]
- The response of the system is excited by an input \( u[k] \) and some initial conditions.
- The difference equations are basically algebraic equations, their solutions can be obtained by direct substitution.
- The solution however is not in closed form and is difficult to develop general properties of the system.
- A number of design techniques have been developed in the z-Transform domain.
1. The z-Transform

Positive and Negative Time Sequence

- A discrete-time signal \( f[k] = f[kT] \), where \( k \) is an integer ranging \((-\infty < k < \infty)\), is called a positive-time sequence if \( f[k] = 0 \) for \( k < 0 \); it is called a negative-time sequence if \( f[k] = 0 \) for \( k > 0 \).
- We mainly consider the positive-time sequences.

z-Transform Pair

- The z-transform is defined as
  \[
  X(z) \equiv Z \{ x[n] \} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}
  \]
  - where \( z \) is a complex variable, called the z-transform variable.

Example

- \( x[n] = \{ 1, 2, 5, 7, 0, 1 \}; \quad x[n] = (1/2)^n u\{n\} \)
0. The z-Transform (c.1)

Example

- \( f[k] = b^k \) for all positive integer \( k \) and \( b \) is a real or complex number.

\[
F(z) = \sum_{n=0}^{\infty} f[n]z^{-n} = \sum_{n=0}^{\infty} b^n z^{-n} = \sum_{n=0}^{\infty} (bz^{-1})^n
\]

- If \( |bz^{-1}| < 1 \), then the infinity power series converges and

\[
F(z) = \frac{1}{1 - bz^{-1}} = \frac{z}{z - b}
\]

- The region \( |b| < |z| \) is called the region of convergence.

Unit Step Sequence

- The unit sequence is defined as

\[
q[n] = \begin{cases} 
1 & \text{for } n = 0, 1, 2, \\
0 & \text{for } n < 0
\end{cases}
\]

- The z-Transform is

Expontential Sequence

- \( f[k] = e^{akT} \)

\[
F(z) = \sum_{n=0}^{\infty} e^{anT} z^{-n} = \frac{1}{1 - e^{aT}}
\]
1. The z-Transform (c.2)

**Region of Convergence**
- For any given sequence, the set of values of z for which the z-transform converges is called the region of convergence.

**Viewpoints**
- The representation of the complex variable z
  \[ z = r e^{j\omega} \]
- Consider the z-transform
  \[ X( r e^{j\omega} ) = \sum_{n=-\infty}^{\infty} x[n] (r e^{j\omega})^{-n} \]
- Convergent Condition
  \[ \sum_{n=-\infty}^{\infty} |x[n]| r^{-n} < \infty \]

ROC includes the unit circle

**Convergence of the z-Transform**
- The z-transform and its derivatives must be continuous function of z.
1. The z-Transform (c.3)

Rational Function

\[ X(z) = \frac{P(z)}{Q(z)} \]

Ex.

\[ x[n] = a^n u[n] \quad x[n] = -a^n u[-n - 1] \]

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \delta[n] )</td>
<td>1</td>
<td>All ( z )</td>
</tr>
<tr>
<td>2. ( u[n] )</td>
<td>( \frac{1}{1 - z^{-1}} )</td>
<td>(</td>
</tr>
<tr>
<td>3. (-u[-n - 1])</td>
<td>( \frac{1}{1 - z^{-1}} )</td>
<td>(</td>
</tr>
<tr>
<td>4. ( \delta[n - m] )</td>
<td>( z^{-m} )</td>
<td>All ( z ) except 0 (if ( m &gt; 0 )) or ( \infty ) (if ( m &lt; 0 ))</td>
</tr>
<tr>
<td>5. ( a^n u[n] )</td>
<td>( \frac{1}{1 - az^{-1}} )</td>
<td>(</td>
</tr>
<tr>
<td>6. (-a^n u[-n - 1])</td>
<td>( \frac{1}{1 - az^{-1}} )</td>
<td>(</td>
</tr>
<tr>
<td>7. ( na^n u[n] )</td>
<td>( \frac{az^{-1}}{(1 - az^{-1})^2} )</td>
<td>(</td>
</tr>
<tr>
<td>8. (-na^n u[-n - 1])</td>
<td>( \frac{az^{-1}}{(1 - az^{-1})^2} )</td>
<td>(</td>
</tr>
<tr>
<td>9. ([\cos \omega_0 n] u[n])</td>
<td>( \frac{1 - [\cos \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}} )</td>
<td>(</td>
</tr>
<tr>
<td>10. ([\sin \omega_0 n] u[n])</td>
<td>( \frac{[\sin \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}} )</td>
<td>(</td>
</tr>
<tr>
<td>11. ([r^n \cos \omega_0 n] u[n])</td>
<td>( \frac{[r \cos \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}} )</td>
<td>(</td>
</tr>
<tr>
<td>12. ([r^n \sin \omega_0 n] u[n])</td>
<td>( \frac{[r \sin \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}} )</td>
<td>(</td>
</tr>
<tr>
<td>13. (\begin{cases} a^n, &amp; 0 \leq n \leq N - 1, \ 0, &amp; \text{otherwise} \end{cases})</td>
<td>( \frac{1 - a^N z^{-N}}{1 - az^{-1}} )</td>
<td>(</td>
</tr>
</tbody>
</table>
2. Properties of the Region of Convergence for the z-Transform

Properties

- The ROC is a ring or disk in the z-plane centered at the origin, i.e., \( 0 < r_R < |z| < r_L \leq \infty \).
- The Fourier transform of \( x[n] \) converges absolutely if and only if the ROC of the z-transform of \( x[n] \) includes the unit circle.
- The ROC cannot contain any poles.
- If \( x[n] \) is a finite-duration sequence, i.e. a sequence that is zero except in a finite interval \( -\infty < N_1 \leq n \leq N_2 \leq \infty \), then the ROC is the entire z-plane except possibly \( z=0 \) or \( z= \infty \).
- If \( x[n] \) is a right-sided sequence, i.e. a sequence that is zero for \( n<N_1<\infty \), the ROC extends outward from the outermost finite pole in \( X(z) \) to \( z= \infty \).
- If \( x[n] \) is a left-sided sequence, i.e., a sequence that is zero for \( n>N_2>-\infty \), the ROC extends inward from the innermost (smallest magnitude) nonzero pole in \( X(z) \) to (and possibly including) \( z=0 \).
- A two-sided sequence is an infinite-duration sequence that is neither right-sided nor left-sided. If \( x[n] \) is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole, and, consistent with property 3, not containing any poles.

The ROC must be a connected region.
2. Properties of the Region of Convergence for the z-Transform

Example

- ROC is a Ring
- ROC is the interior of a circle
- ROC is the exterior of a circle

No common ROC case?
3. The Inverse z-Transform

Methods

- Direct Division
- Partial Fraction Expansion

Direct Division

\[ F(z) = -2z^2 + 3z + 3z^{-2} + 3z^{-3} + 9z^{-4} \]

\[ f[k] = \{-2, 3, 0, 0, 3, 3, 9, \ldots\} \]

Ex. \( 3/(z^2 - z - 2) \)

\[
F(z) = \frac{-2z^4 + 5z^3 + z^2 - 6z + 3}{z^2 - z - 2}
\]

\[
\begin{align*}
-2z^2 + 3z & + 3z^{-2} + 3z^{-3} + 9z^{-4} \\
-2z^4 + 5z^3 + z^2 - 6z + 3 & \\
-2z^4 + 2z^3 + 4z^2 & \\
3z^3 - 3z^2 - 6z + 3 & + 3
\end{align*}
\]
3. The Inverse z-Transform (c.1)

Partial Fraction Expansion and Table Lookup

\[ X(z) = \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} \]

- If \( M < N \) and the poles are all first order
  \[ X(z) = \frac{b_0}{a_0} \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})} \]
  \[ A_k = (1 - d_k z^{-1}) X(z) \bigg|_{z=d_k} \]

- If \( M \geq N \) and the poles are all first order, the complete partial fraction expression can be
  \[ X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})} \]
  \[ A_k = (1 - d_k z^{-1}) X(z) \bigg|_{z=d_k} \]

- If \( X(z) \) has multiple-order poles and \( M \geq N \)
  \[ X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k \neq i}^{N} \frac{A_k}{(1 - d_k z^{-1})} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m} \]
  \[ C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[ (1 - d_i w)^s X(w^{-1}) \right] \right\}_{w=d_i^{-1}} \]
3. The Inverse $z$-Transform (c.2)

### Examples

$$ X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{1 + 2z^{-1} + z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}} $$

- **ROC:** $|z| > 1$

- **ROC:** $|z| < \frac{1}{2}$

- **ROC:** $\frac{1}{2} < |z| < 1$
4. The Unilateral $z$-Transform

**Definition**

$$X(z) = Z[x[n]] = \sum_{n=0}^{\infty} x[n] z^{-n}$$

**Time Delay**

$$x[n] \quad \circ \quad X(z)$$

$$x[n - k] \quad \circ \quad z^{-k} X(z) + \sum_{n=1}^{k} x[-n] z^{-k+n}$$

$$x[n + k] \quad \circ \quad z^{i} \left[ X(z) - \sum_{n=0}^{k-1} x[n] z^{-n} \right]$$
5. Solving the Difference Equations

Goals

- Solving LTI L differential equations using Laplace transform.

Example

\[ 2y[k] + 3y[k-1] + y[k-2] = u[k] + u[k-1] - u[k-2] \]

where \( u(t) = q(t) \), \( y(-1) = -1 \) and \( y(-2) = 1 \).

Exercise

Find the response of

\[ y[k+1] - 2y[k] = u[k] \quad \text{and} \quad y[k+1] - 2y[k] = u[k+1] \]

\( y[-1] = 1 \) and \( u[k] = 1 \), for \( k = 0, 1, 2, \ldots \).
6. Zero-Input Response-- Characteristic Polynomial

Consider the zero-input response

\[ Y_{zi}(z) = \frac{-3y[-1] - y[-2] - y[-1]z^{-1}}{2 + 3z^{-1} + z^{-2}} = \frac{(-3y[-1] - y[-2])z^2 - y[-1]z}{2z^2 + 3z + 1} \]

- The denominator of \( Y_{zi} \) is called the characteristic polynomial.
- The roots of \( Y_{zi} \) is called the modes of the system.

Why the name, "mode"?

- The zero-input response of the system excited by any initial conditions can always be expressed as

\[ Y_{zi}(z) = \frac{(-3y[-1] - y[-2])z^2 - y[-1]z}{2z^2 + 3z + 1} = \frac{k_1 z}{z + 0.5} + \frac{k_2 z}{z + 1} \]

- The zero-input response is for \( t \geq 0 \)

\[ y_{zi}(k) = k_1 (-0.5)^k + k_2 (-1)^k \]

The form of the zero-input response excited by any initial conditions is completely determined by the modes of the system.

The zero-input response is always a linear combination of the two time functions \((-0.5)^k\) and \((-1)^k\).
6. Zero-State Response--- Transfer Function

Consider

\[ 2y[k] + 3y[k-1] + y[k-2] = u[k] + u[k-1] - u[k-2] \]

If all initial conditions are zero, we have

\[ Y(z) = \frac{1 + z^{-1} - z^{-2}}{2 + 3z^{-1} + z^{-2}} U(z) \]
\[ U(z) = \frac{z^2 + z - 1}{2z^2 + 3z + 1} U(z) \]

Ways to Find Transfer Functions

- The transfer function is the z transform of the impulse response.
- The function can be obtained from the zero-state response excited by any input, in particular, step or sinusoidal functions.
- The function can be obtained from the difference equation description.

The transfer function of the above example is \((z^2 + z - 1)/(2z^2 + 3z +1)\)

The Transfer Functions

The ratio of the z transforms of the output and input with all initial conditions zero or

\[ H(z) = \frac{Y(z)}{U(z)} \bigg|_{\text{initial conditions } = 0} \]

The transfer function of the above example is \((z^2 + z - 1)/(2z^2 + 3z +1)\)
6.1 Poles and Zeros of Proper Transfer Functions

For a proper rational functions

\[ H(z) = \frac{N(z)}{D(z)} \]

where \(N(z)\) and \(D(z)\) are polynomials with real coefficients. If \(N(z)\) and \(D(z)\) have no common factors, then the roots of \(D(z)\) and \(N(z)\) are respectively the poles and zeros of the system.

**Examples**

\[ Y(z) = \frac{3(z + 4)}{2(z + 0.5)(z + 1)} U(z) \]

**Definition**

A finite real or complex number \(\lambda\) is a pole of \(H(z)\) if the absolute value of \(H(\lambda) = \ast\). It is a zero of \(H(z)\) if \(H(\lambda) = 0\).

**Examples**

Find the zero-state response of a system with transfer function, \(H(z) = \frac{(z^2 + z - 1)}{(2z^2 + 3z + 1)}\) excited by unit step function.
6.1 Poles and Zeros of Proper Transfer Functions (c.1)

\textbf{Consider the following systems}

\[ H_1(z) = \frac{0.551}{(z + 0.9)(z - 0.8 + j0.5)(z - 0.8 - j0.5)} = \frac{0.551}{z^3 - 0.7z^2 - 0.55z + 0.801} \]

\[ H_2(z) = \frac{z - 1}{z^3 - 0.7z^2 - 0.55z + 0.801} \]

\[ H_3(z) = \frac{5.51(z - 0.9)}{z^3 - 0.7z^2 - 0.55z + 0.801} \]

\[ H_4(z) = \frac{-5.51(z - 1.1)}{z^3 - 0.7z^2 - 0.55z + 0.801} \]

\[ H_5(z) = \frac{5.51(z^2 - 1.9z + 1)}{z^3 - 0.7z^2 - 0.55z + 0.801} \]

\textbf{Positions of zeros and poles}
6.2 Time Responses of Modes and Poles

Remarks

- The zero-input response is essentially dictated by the modes; the zero-state response is essentially dictated by the poles.

Three parts in z-plane

- The unit circle
- The region outside the unit circle.
- The region inside the unit circle.

Observations

- The poles $\sigma + j\omega$ or $re^{+j\theta}$
- The response $r^k \cos k\theta$ or $r^k \sin k\theta$
- $\theta$ determines the frequency of the oscillation.
- The highest frequency is determined by $(-\pi/T, \pi/T)$. 

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6.2 Time Responses of Modes and Poles (c.1)

**Summary**

- The time response of a pole (mode), simple or repeated, approaches zero as $k \to \infty$ if and only if the pole (mode) lies inside the unit circle or its magnitude is smaller than 1.

- The time response of a pole (mode) approaches a nonzero constant as $k \to \infty$ if and only if the pole is simple and located at the $z=1$.

**Examples**

- $y[k+3] - 1.6y[k+2] - 0.76y[k+1] - 0.08y[k] = u[k+2] - 4u[k]$.

**Table 5.3 Time responses of poles and modes as $k \to \infty$**

<table>
<thead>
<tr>
<th>Poles or modes</th>
<th>Simple $(n = 1)$</th>
<th>Repeated $(n = 2, 3, \ldots)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inside the unit circle</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Outside the unit circle</td>
<td>$\pm\infty$</td>
<td>$\pm\infty$</td>
</tr>
<tr>
<td>$1/(z - 1)^n$</td>
<td>A constant</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$1/[(z - e^{j\theta})(z - e^{-j\theta})]^n$</td>
<td>A sustained oscillation</td>
<td>$\pm\infty$</td>
</tr>
</tbody>
</table>
7. Transfer-Function Representation-- Complete Characterization

**System Description for LTIL systems**
- Convolution
- Difference equation
- Transfer function.

**System Connection**
- The transfer function of connection can be easily derived from algebraic manipulation of transfer functions.
7. Transfer-Function Representation-- Complete Characterization (c.1)

**Questions**

- **Transfer functions represent the input-output relation when initial conditions are set to zeros.**
- **What is the representation of the transfer function for a system?**

**Consider the difference equations**

- **D(z) Y(z) = N(z) U (z)**
  
  Ex. \( D(z) = (z+0.5)(z-1)(z-2) \); \( N(z) = z^2 - 3z + 2 \)
  
  \[ y_{zs}[k] = k_1(-0.5)^k + \text{(terms due to the poles of } U(z)) \]

**Missing Poles**

- If \( N(z) \) and \( D(z) \) have common factors, \( R(z) \), then the roots of \( R(z) \) are modes but not poles of the system and
  
  \{the set of the poles\} \ \ {the set of the modes\}

- The root of \( R(z) \) are called the missing poles.

**Completed Characterization**

- A system is completed characterized by its transfer function if \( N(z) \) and \( D(z) \) are coprime.