Lossy Compression: Math Basics

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Definitions

- **Lossless compression:** $x = x'$
  - A.k.a. *entropy coding, reversible coding*

- **Lossy compression:** $x \neq x'$
  - A.k.a. *irreversible coding*
Motivation

- Why use lossy compression?
  - Fundamental limits on lossless compression
  - Human cognitive apparatus
    - Fundamental limits on human perception
      - If you can’t see/hear it, why encode it?
    - Abilities to recover from partial loss
      - E.g. lower frame rate will make movement jerky but perceptible
        (think video via satellite phone)
  - Many sources have very high “natural” bit rates
    - Audio/video/etc.
Measures of Difference Distortion

- **Notation**
  - \( \{x_n\} \rightarrow \text{original source output} \)
  - \( \{y_n\} \rightarrow \text{reconstructed output} \)

- **Squared error**
  - \( d(x, y) = (x - y)^2 \)

- **Absolute difference**
  - \( d(x, y) = |x - y| \)
Measures of Difference Distortion (2)

- **Mean squared error** (MSE)

\[ \sigma^2 = \frac{1}{N} \sum_{n=1}^{N} (x_n - y_n)^2 \]

- **Signal-to-noise ratio** (SNR)

\[ \text{SNR} = \frac{\sigma_x^2}{\sigma_d^2} \]

- **Decibel** (dBel)

\[ \text{SNR(dB)} = 10 \log_{10} \left( \frac{\sigma_x^2}{\sigma_d^2} \right) \]
Measures of Difference Distortion (3)

- **Peak-signal-to-noise ratio** (PSNR)

\[
\text{PSNR}(\text{dB}) = 10 \log_{10} \frac{x_{\text{peak}}^2}{\sigma_d^2}
\]

- **Average absolute difference**

\[
\sigma^2 = \frac{1}{N} \sum_{n=1}^{N} |x_n - y_n|
\]

- **Absolute maximum error**

\[
d_\infty = \max_n |x_n - y_n|
\]
Eye Physiology

- Conea
- Lens
- Retina
  - Cones
  - Rods
- Outer Synaptic Layer
- Bipolar Cells
- Inner Synaptic Layer
- Ganglion Cell
Eye Physiology (c.1)

Visual Pathways

- Superior oblique: Moves eye upward, inward, and counterclockwise
- Medial rectus: Moves eye inward
- Inferior oblique: Moves eye upward, outward, and clockwise
- Lateral rectus: Moves eye outward
- Inferior rectus: Moves eye downward, inward, and clockwise

An eye looking at an object

Microsoft Illustration
Eye Physiology (c.2)

Two Types of Photoreceptors in Retina

- **Rods**
  - About 100 million in number
  - Relatively long and thin
  - Provides scotopic vision or dim-light vision

- **Cones**
  - About 6.5 million in number
  - Shorter and thicker
  - Provide photopic vision or bright-light vision
  - Highly sensitive to color
  - Densely packed in the center of Retina (called fovea)
Visual Phenomena

- **Contrast Sensitivity**
  - Sensitive to luminance contrast rather than the absolute luminance values.
  - Weber’s Ratio
    \[ \frac{\Delta I}{I} = \text{constant} = d[\log(I)] \]
  - Related to surrounding and background luminance
Visual Phenomena -- Contrast Sensitivity

- Logarithmic Law
- Power Law
- Background Ratio

**TABLE 3.1** Luminance to Contrast Models

<p>| | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>Logarithmic law</td>
<td>( c = 50 \log_{10} f, \ 1 \leq f \leq 100 )</td>
</tr>
</tbody>
</table>
| 2 | Power law | \( c = \alpha_n f^{\frac{1}{n}}, \ n = 2, 3, \ldots \)  
    | | \( \alpha_2 = 10, \alpha_3 = 21.9 \) |
| 3 | Background ratio | \( c = \frac{f(f_B + 100)}{f_B + f} \)  
    | | \( f_B = \text{background luminance} \) |

The luminance \( f \) lies in the interval \([0, 100]\) except in the logarithmic law. Contrast scale is over \([0, 100]\).
Visual Phenomena-- Mach Band

- **Overshoot**
  - Uniform luminance in the strip
  - The visual appearance is darker at its right side than its left.

![Ramp chart photo](image)

![Ramp chart intensity distribution](image)
Visual Phenomena -- MTF of the System

- **The Frequency Response**
  - The peak varies with the viewer and generally lies between 3 and 10 cycles/degree.
  - The contrast sensitivity also depends on the orientation of the grating, being maximum for horizontal and vertical gratings.
  - The angular sensitivity variations are within 3 dB (maximum deviation is at 45°).
  - The curve fitting procedure has yielded.
    - ex. $A=2.6, \alpha = 0.0192, \beta=1.1$, and $\rho_0=(0.114)^{-1} = 8.772$
    - $H(\xi_2, \xi_2) = H_\rho(\rho) = A[\alpha + (\frac{\rho}{\rho_0})]\exp[-(\frac{\rho}{\rho_0})^\beta]$
    - where $\rho = \sqrt{\xi_1^2 + \xi_2^2}$ cycles/degree
Probability Models

- **Lossless approach**
  - Based on empirical probability distributions
  - … we needed an exact match

- **Lossy compression**
  - We can afford to *approximate* the actual distribution using a ‘nice’ analytical model

- **Advantages**
  - Use the analytical properties to improve
    - Compression
    - Reconstruction
**Probability Models (2)**

- **Uniform distribution**
  - Ignorance model—all values equally probable
    $$f_X = \begin{cases} 
    \frac{1}{b-a} & \text{for } a \leq x \leq b \\
    0 & \text{otherwise} 
    \end{cases}$$

- **Gaussian distribution**
  - Very common model
    - Analytically tractable
    - Limiting distribution turns Gaussian
    $$f_X = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
Laplacian distribution
- Most of the weight of the pdf is around the mean (0)

\[
f_X = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{-\sqrt{2}|x|}{\sigma}}
\]

Gamma distribution
- Even more concentrated around the mean (0)

\[
f_X = \frac{4\sqrt{3}}{\sqrt{8\pi\sigma^4} |x|} e^{-\frac{-\sqrt{3}|x|}{2\sigma}}
\]
Probability Models Comparison

\[ \mu = 0, \sigma^2 = 1 \]
Linear System Models

- AutoRegressive Moving Average—ARMA($N,M$)
  \[ x_n = \sum_{i=1}^{N} a_i x_{n-i} + \sum_{j=1}^{M} b_j \varepsilon_{n-j} + \varepsilon_n \]

- Autoregressive—AR($N$) == ARMA($N,0$)
  \[ x_n = \sum_{i=1}^{N} a_i x_{n-i} + \varepsilon_n \]

- \[ P(x_n | x_{n-1}, x_{n-2}, \ldots) = P(x_n | x_{n-1}, x_{n-2}, \ldots, x_{n-N}) \]

- \[ R_{\varepsilon\varepsilon}(k) = \begin{cases} \sigma_\varepsilon^2 & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases} \]
AR(1) Autocorrelation Function
Example AR(1) Processes

\[ a_1 = 0.99 \]

\[ a_1 = 0.60 \]
Example AR(1) Processes (2)

\[ a_1 = -0.99 \]

\[ a_1 = -0.60 \]
Negative Autocorrelation Function

The graph shows the autocorrelation function $R(k)$ for two different values of $a_1$. The solid line represents $a_1 = -0.99$ and the dashed line represents $a_1 = -0.6$. The y-axis represents $R(k)$ with values ranging from -1.0 to 1.0, and the x-axis represents $k$ with values from 0 to 20.
Homeworks & References

- Problems

- References