Mathematical Preliminaries for Transforms, Subbands, and Wavelets

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Reading Assignment

  - You should know this already
- Topics:
  - Dot/inner product
  - Vector space
  - Subspace
  - Basis
  - Orthogonal & orthonormal sets
Vector Space

**DEFINITION**

Let $V$ be a set on which the operations of addition and scalar multiplication are defined. By this we mean that, with each pair of elements $x$ and $y$ in $V$, we can associate a unique element $x + y$ that is also in $V$, and with each element $x$ in $V$ and each scalar $a$, we can associate a unique element $ax$ in $V$. The set $V$ together with the operations of addition and scalar multiplication is said to form a **vector space** if the following axioms are satisfied.

A1. $x + y = y + x$ for any $x$ and $y$ in $V$.
A2. $(x + y) + z = x + (y + z)$ for any $x$, $y$, $z$ in $V$.
A3. There exists an element $0$ in $V$ such that $x + 0 = x$ for each $x$ in $V$.
A4. For each $x$ in $V$, there exists an element $-x$ in $V$ such that $x + (-x) = 0$.
A5. $a(x + y) = ax + ay$ for each scalar $a$ and any $x$ and $y$ in $V$.
A6. $(a + b)x = ax + bx$ for any scalars $a$ and $b$ and any $x$ in $V$.
A7. $(ab)x = a(bx)$ for any scalars $a$ and $b$ and any $x$ in $V$.
A8. $1 \cdot x = x$ for all $x$ in $V$.

**C1.** If $x \in V$ and $\alpha$ is a scalar, then $\alpha x \in V$.

**C2.** If $x$, $y \in V$, then $x + y \in V$. 
Vector Space-- Example

Let $C[a, b]$ denote the set of all real-valued functions that are defined and continuous on the closed interval $[a, b]$. In this case our universal set is a set of functions. Thus our vectors are the functions in $C[a, b]$. The sum $f + g$ of two functions in $C[a, b]$ is defined by

$$(f + g)(x) = f(x) + g(x)$$

for all $x$ in $[a, b]$. The new function $f + g$ is an element of $C[a, b]$, since the sum of two continuous functions is continuous. If $f$ is a function in $C[a, b]$ and $\alpha$ is a real number, define $\alpha f$ by

$$(\alpha f)(x) = \alpha f(x)$$
**Subspace and Basis**

**Definition**

If $S$ is a nonempty subset of a vector space $V$, and $S$ satisfies the following conditions:

(i) $\alpha x \in S$ whenever $x \in S$ for any scalar $\alpha$
(ii) $x + y \in S$ whenever $x \in S$ and $y \in S$

then $S$ is said to be a **subspace** of $V$.

**Definition**

The vectors $v_1, v_2, \ldots, v_n$ form a **basis** for a vector space $V$ if and only if

(i) $v_1, \ldots, v_n$ are linearly independent.
(ii) $v_1, \ldots, v_n$ span $V$.

**Definition**

The vectors $v_1, v_2, \ldots, v_n$ in a vector space $V$ are said to be **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$
**Inner Product**

**Definition**

An **inner product** on a vector space $V$ is an operation on $V$ that assigns to each pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $V$ a real number $\langle \mathbf{x}, \mathbf{y} \rangle$ satisfying the following conditions:

I. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ with equality if and only if $\mathbf{x} = \mathbf{0}$.

II. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all $\mathbf{x}$ and $\mathbf{y}$ in $V$.

III. $\langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $V$ and all scalars $\alpha$ and $\beta$.

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**The Vector Space $\mathbb{R}^n$**

The standard inner product for $\mathbb{R}^n$ is the scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$$

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**The Vector Space $\mathbb{C}[a, b]$**

In $\mathbb{C}[a, b]$ we may define an inner product by

$$\langle f, g \rangle = \int_{a}^{b} f(x)g(x) \, dx$$

---

**The Vector Space $\mathbb{R}^{m \times n}$**

Given $A$ and $B$ in $\mathbb{R}^{m \times n}$, we can define an inner product by

$$\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$$

---

Given a vector $\mathbf{w}$ with positive entries, we could also define an inner product on $\mathbb{R}^n$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i w_i$$

The entries $w_i$ are referred to as weights.
Example 2  For the vector space $C[-\pi, \pi]$, if we use a constant weight function $w(x) = 1/\pi$ to define an inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx$$

then it follows that

$$\langle \cos x, \sin x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \sin x \, dx = 0$$

$$\langle \cos x, \cos x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \cos x \, dx = 1$$

$$\langle \sin x, \sin x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \sin x \, dx = 1$$

Thus $\cos x$ and $\sin x$ are orthogonal unit vectors with respect to this inner product. It follows from the Pythagorean Law that

$$\| \cos x + \sin x \| = \sqrt{2}$$
Orthogonal and Orthonormal Sets

**DEFINITION**

Let \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) be nonzero vectors in an inner product space \( V \). If \( \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \) whenever \( i \neq j \), then \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \) is said to be an **orthogonal set** of vectors.

**Example 3** In \( C[-\pi, \pi] \) with inner product

\[
\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx
\]

the set \( \{1, \cos x, \cos 2x, \ldots, \cos nx\} \) is an orthogonal set of vectors, since for any positive integers \( j \) and \( k \)

\[
\langle 1, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx \, dx = 0
\]

\[
\langle \cos jx, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos jx \cos kx \, dx = 0 \quad (j \neq k)
\]

The functions \( \cos x, \cos 2x, \ldots, \cos nx \) are already unit vectors since

\[
\langle \cos kx, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 kx \, dx = 1 \quad \text{for} \quad k = 1, 2, \ldots, n
\]

To form an orthonormal set, we need only find a unit vector in the direction of \( 1 \).

\[
\|1\|^2 = \langle 1, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \, dx = 2
\]

Thus \( 1/\sqrt{2} \) is a unit vector and hence \( \{1/\sqrt{2}, \cos x, \cos 2x, \ldots, \cos nx\} \) is an orthonormal set of vectors.
**Example 2** We saw in Example 1 that, if \( \mathbf{v}_1 = (1, 1, 1)^T \), \( \mathbf{v}_2 = (2, 1, -3)^T \), and \( \mathbf{v}_3 = (4, -5, 1)^T \), then \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is an orthogonal set in \( \mathbb{R}^3 \). To form an orthonormal set, let

\[
\mathbf{u}_1 = \left( \frac{1}{\|\mathbf{v}_1\|} \right) \mathbf{v}_1 = \frac{1}{\sqrt{3}} (1, 1, 1)^T
\]

\[
\mathbf{u}_2 = \left( \frac{1}{\|\mathbf{v}_2\|} \right) \mathbf{v}_2 = \frac{1}{\sqrt{14}} (2, 1, -3)^T
\]

\[
\mathbf{u}_3 = \left( \frac{1}{\|\mathbf{v}_3\|} \right) \mathbf{v}_3 = \frac{1}{\sqrt{42}} (4, -5, 1)^T
\]

**Example 3** In \( C[-\pi, \pi] \) with inner product

\[
(2) \quad \langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx
\]

the set \( \{1, \cos x, \cos 2x, \ldots, \cos nx\} \) is an orthogonal set of vectors, since for any positive integers \( j \) and \( k \)

\[
\langle 1, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx \, dx = 0
\]

\[
\langle \cos jx, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos jx \cos kx \, dx = 0 \quad (j \neq k)
\]

The functions \( \cos x, \cos 2x, \ldots, \cos nx \) are already unit vectors since

\[
\langle \cos kx, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 kx \, dx = 1 \quad \text{for} \quad k = 1, 2, \ldots, n
\]

To form an orthonormal set, we need only find a unit vector in the direction of 1.

\[
\|1\|^2 = (1, 1) = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \, dx = 2
\]

Thus \( 1/\sqrt{2} \) is a unit vector and hence \( \{1/\sqrt{2}, \cos x, \cos 2x, \ldots, \cos nx\} \) is an orthonormal set of vectors.
Orthonormal Sets

**Theorem 5.5.2** Let \( \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \) be an orthonormal basis for an inner product space \( V \).

If \( \mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{u}_i \), then \( c_i = \langle \mathbf{v}, \mathbf{u}_i \rangle \).

**Proof**

\[
\langle \mathbf{v}, \mathbf{u}_i \rangle = \left( \sum_{j=1}^{n} c_j \mathbf{u}_j, \mathbf{u}_i \right) = \sum_{j=1}^{n} c_j \langle \mathbf{u}_j, \mathbf{u}_i \rangle = \sum_{j=1}^{n} c_j \delta_{ji} = c_i
\]

As a consequence of Theorem 5.5.2 we can state two more important results.

**Corollary 5.5.3** Let \( \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \) be an orthonormal basis for an inner product space \( V \). If \( \mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{u}_i \) and \( \mathbf{v} = \sum_{i=1}^{n} b_i \mathbf{u}_i \), then

\[
\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} a_i b_i
\]

**Proof** By Theorem 5.5.2

\[
\langle \mathbf{v}, \mathbf{u}_i \rangle = b_i \quad i = 1, \ldots, n
\]

Therefore,

\[
\langle \mathbf{u}, \mathbf{v} \rangle = \left( \sum_{i=1}^{n} a_i \mathbf{u}_i, \mathbf{v} \right) = \sum_{i=1}^{n} a_i \langle \mathbf{u}_i, \mathbf{v} \rangle = \sum_{i=1}^{n} a_i \langle \mathbf{v}, \mathbf{u}_i \rangle = \sum_{i=1}^{n} a_i b_i
\]
5. Orthonormal Sets

**Corollary 5.5.4 (Parseval’s Formula)** If \( \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \) is an orthonormal basis for an inner product space \( V \) and \( \mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{u}_i \), then

\[
\|\mathbf{v}\|^2 = \sum_{i=1}^{n} c_i^2
\]

**Proof** If \( \mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{u}_i \), then, by Corollary 5.5.3,

\[
\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^{n} c_i^2
\]

**Example 4** The vectors

\[
\mathbf{u}_1 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T \quad \text{and} \quad \mathbf{u}_2 = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)^T
\]

form an orthonormal basis for \( R^2 \). If \( \mathbf{x} \in R^2 \), then

\[
\mathbf{x}^T \mathbf{u}_1 = \frac{x_1 + x_2}{\sqrt{2}} \quad \text{and} \quad \mathbf{x}^T \mathbf{u}_2 = \frac{x_1 - x_2}{\sqrt{2}}
\]
Fourier Series

- Originally developed in 1812 to study heat diffusion equations
- Given a periodic function $f(t)$ with period $T$:

$$ f(t) = f(t + nT), \quad n = \pm 1, \pm 2, \ldots $$

- Trigonometric form:

$$ f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t, \quad \omega_0 = \frac{2\pi}{T}. $$

- Euler’s identity:

$$ e^{i\phi} = \cos \phi + i \sin \phi, \quad i = \sqrt{-1} $$

- Exponential form:

$$ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} $$
Fourier Series (2)

- **Basis:**
  \[ \{ e^{in\omega_0 t} \} \]

- **Inner product:**
  \[
  \langle f(t), g(t) \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)g(t)^* \, dt, \quad (t_0 = 0)
  \]
  \[
  \langle e^{in\omega_0 t}, e^{im\omega_0 t} \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} e^{in\omega_0 t} e^{-im\omega_0 t} \, dt = \frac{1}{T} \int_{t_0}^{t_0+T} e^{i(n-m)\omega_0 t} \, dt
  \]
Fourier Series (3)

- Let $n = m$:

$$
\langle e^{i n \omega_0 t}, e^{i n \omega_0 t} \rangle = \frac{1}{T} \int_0^T e^{i(n-n) \omega_0 t} \, dt = 1
$$

- Let $k = n - m \neq 0$

$$
\langle e^{i n \omega_0 t}, e^{i m \omega_0 t} \rangle = \frac{1}{T} \int_0^T e^{i k \omega_0 t} \, dt = \frac{1}{i k \omega_0} \left( e^{i k \omega_0 T} - 1 \right)
$$

$$
= \frac{1}{i k \omega_0} \left( e^{i 2k \pi} - 1 \right) = 0
$$

$$
e^{i k 2\pi} = \cos(2k\pi) + i \sin(2k\pi) = 1, \quad \omega_0 = \frac{2\pi}{T}
$$
Fourier Series Advantages

- $f(t)$ (or $f(x)$) represents the signal as a function of time/space
- A Fourier series gives us a **frequency-based** representation:
  - $\{e^{2i\omega t}\}$ fluctuates twice as fast as $\{e^{i\omega t}\}$
  - $\{e^{4i\omega t}\}$ fluctuates twice as fast as $\{e^{2i\omega t}\}$
  - …
- $\{c_n\}$ give us a measure how much of each of the different basis fluctuations are present in the signal
- Recall that 1 cycle per second $= 1$ Hz
Consider the periodic extension of a non-periodic function \( f(t) \):

\[
f_P(t) = \sum_{n=-\infty}^{\infty} f(t - nT), \quad T > t_1
\]

Fourier series expansion:

\[
f_P(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega_0 t}, \quad c_n = \frac{1}{T} \int_{-T/2}^{T/2} f_P(t)e^{-in\omega_0 t} dt
\]
Let $C(n, T) = c_n T$, $\Delta \omega = \omega_0$

$$C(n,T) = \int_{-T/2}^{T/2} f_P(t)e^{-in\Delta \omega t} \, dt$$

$$f_P(t) = \sum_{n=-\infty}^{\infty} \frac{C(n,T)}{T} e^{-in\Delta \omega t}$$

To recover $f(t)$ from $f_P(t)$ we let $T \to \infty$

$$\lim_{T \to \infty} \int_{-T/2}^{T/2} f_P(t)e^{-in\Delta \omega t} \, dt = \int_{-\infty}^{\infty} f_P(t)e^{-i\omega t} \, dt$$
Fourier Transform (3)

\[ f(t) = \lim_{T \to \infty} f_p(t) = \lim_{\Delta \omega \to 0} \sum_{n=-\infty}^{\infty} C(n, T) \frac{\Delta \omega}{2\pi} e^{-in\Delta \omega t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} \, dt \]

Fourier Transform

\[ F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \]

Inverse Fourier Transform

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} \, dt \]
FT Properties: Perseval’s Theorem

- **Energy preservation:**

\[
\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega
\]

Note: \(1/2\pi\) factor is a result of Hz-to-radians conversion
FT Properties: Modulation

\[ \mathcal{F}[f(t)e^{-i\omega t}] = F(\omega - \omega_0) \]

- **Justification:**

\[
\cos(\omega_0 t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}
\]

\[
\mathcal{F}[f(t)\cos(\omega_0 t)] = \frac{1}{2}(F(\omega - \omega_0) + F(\omega + \omega_0))
\]
FT Properties: **Convolution Theorem**

- Convolution

\[ f(t) = f_1(t) \otimes f_2(t) \equiv \int_{-\infty}^{\infty} f_1(t) f_2(t - \tau) d\tau \]

- Theorem

\[ F(\omega) = F_1(\omega) F_2(\omega) \]

\[ F(\omega) = \mathcal{F}[f(t)] = \mathcal{F}[f_1(t) \otimes f_2(t)] \]

\[ F_1 = \mathcal{F}[f_1(t)], \quad F_2 = \mathcal{F}[f_2(t)] \]
Linear Systems

- Let \( L \) be a linear system with input \( f(t) \) and output \( g(t) \):
  \[
g(t) = L[f(t)]
  \]

- Properties
  - Homogeneity
    \[
    L[f_1(t) + f_2(t)] = L[f_1(t)] + L[f_2(t)]
    \]
  - Scaling
    \[
    L[\alpha f(t)] = \alpha L[f(t)]
    \]
  - Together known as **superposition**
Linear Systems (2)

- We are interested in *time-invariant* systems:
  \[ L[f(t - t_0)] = g(t - t_0) \]

- Scaling & delay
  \[ L[\cos(\omega_0 t)] = \alpha \cos(\omega_0(t - t_d)) \]
  or
  \[ L[e^{i\omega_0 t}] = \alpha e^{i\omega_0 (t - t_d)} \]

- Phase:
  \[ \phi = \omega_0 t_d \]
Gain function: $\alpha(\omega)$

Phase function: $\phi(\omega)$

Transfer function

$$H(\omega) = \alpha(\omega)e^{i\phi(\omega)}$$

Convolution

$$G(\omega) = H(\omega)F(\omega)$$
Impulse Response

\[ f_s(t) = \sum f(n\Delta t) \text{rect} \left( \frac{t - n\Delta t}{\Delta t} \right) \]

\[ \text{rect} \left( \frac{t}{T} \right) = \begin{cases} 
1 & |t| < \frac{T}{2} \\
0 & \text{otherwise.} 
\end{cases} \]
Impulse Response (2)

\[ L[f_S(t)] = L \left[ \sum f(n\Delta t) \text{rect} \left( \frac{t - n\Delta t}{\Delta t} \right) \right] \]

\[ = L \left[ \sum f(n\Delta t) \frac{\text{rect} \left( \frac{t - n\Delta t}{\Delta t} \right)}{\Delta t} \right] \Delta t. \]

by superposition:

\[ L[f_S(t)] = \sum f(n\Delta t) L \left[ \frac{\text{rect} \left( \frac{t - n\Delta t}{\Delta t} \right)}{\Delta t} \right] \Delta t. \]
Impulse Response (3)

\[ L[f_s(t)] = \sum f(n\Delta t) L\left[ \frac{\text{rect}(\frac{t-n\Delta t}{\Delta t})}{\Delta t} \right] \Delta t. \]

Let \( t \to 0 \)

\[ \lim_{\Delta t \to 0} \frac{\text{rect}(\frac{t-n\Delta t}{\Delta t})}{\Delta t} = \delta(t). \]

**Dirac delta function**

\[ L[f(t)] = \lim_{\Delta t \to 0} L[f_s(t)] = \int f(\tau) L[\delta(t-\tau)] d\tau. \]
Impulse Response (4)

\[ L[f(t)] = \lim_{\Delta t \to 0} L[f_s(t)] = \int f(\tau) L[\delta(t - \tau)] d\tau. \]

\[ h(t) = L[\delta(t)]. \]

**Impulse response function**

Assuming time-invariance:

\[ L[f(t)] = \int f(\tau) h(t - \tau) d\tau. \]

\[ H(\omega) = \mathcal{F}[h(t)] \]

Delta function sifting property:

\[ \int_{t_1}^{t_2} f(t) \delta(t - t_0) dt = \begin{cases} f(t_0) & t_1 \leq t_0 \leq t_2 \\ 0 & \text{otherwise.} \end{cases} \]
Filters

- Filter:
  - A linear system that permits certain frequency components to pass through while attenuating the rest

- Low-pass filters
  - Allows only components below frequency $W$ Hz
  - Ideal low-pass filter:

\[
H(\omega) = \begin{cases} 
  e^{-i\alpha\omega} & |\omega| < 2\pi W \\
  0 & \text{otherwise}
\end{cases}
\]

- Bandwidth $W$

- Similarly
  - High-pass filter
  - Band-pass filter:
Filter Examples

**Ideal low-pass**

\[ |H(\omega)| \]

\[ \begin{array}{c}
\omega = 0 \\
\omega = 2\pi W \\
\end{array} \]

**Ideal high-pass**

\[ |H(\omega)| \]

\[ \begin{array}{c}
\omega = 0 \\
\omega = 2\pi W \\
\end{array} \]

**Real low-pass**

\[ |H(\omega)| \]

\[ \begin{array}{c}
\omega = 0 \\
\omega = 2\pi W \\
\end{array} \]

**Ideal band-pass**

\[ |H(\omega)| \]

\[ \begin{array}{c}
\omega = \omega_0 \\
\omega = \omega_0 + 2\pi W \\
\end{array} \]
Ideal Sampling: Frequency View

\[ F_p(\omega) = \sum_{n=\infty}^{\infty} F(\omega - n\sigma_0), \quad \sigma_0 = 4\pi W. \]
Ideal Sampling: Frequency View (2)

- Fourier series expansion
  \[ F_p(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in}{2W} \omega t} \]

- Coefficients through Inner Products
  \[ c_n = \frac{1}{4\pi W} \int_{-2\pi W}^{2\pi W} F_p(\omega) e^{-in \frac{1}{2W} \omega t} d\omega \]

- Hence
  \[ c_n = \frac{1}{2W} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i \frac{n}{2W} \omega t} d\omega \right] \]

Knowing \( \{c_n\}_{n=-\infty}^{\infty} \) and the value of \( W \), we can reconstruct \( F_p(\omega) \).
Ideal Sampling: Frequency View (3)

- Reconstruct $f(t)$ from $c_n$.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F(\omega)e^{-i\omega t} d\omega$$

$$f(t) = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F_p(\omega)e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} \sum_{n=-\infty}^{\infty} c_n e^{in\frac{\omega}{2W}} e^{-i\omega t} d\omega$$

$$Sinc[x] = \frac{\sin(\pi x)}{\pi x}$$

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n \int_{-2\pi W}^{2\pi W} e^{i\left(t - \frac{n}{2W}\right)\omega} d\omega$$

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) Sinc\left[2W\left(t - \frac{n}{2W}\right)\right]$$

Sample of $x_g(t)$
Ideal Sampling—Time Domain View

Let us look at this process from a slightly different point of view, starting with the sampling operation. Mathematically, we can represent the sampling operation by multiplying the function $f(t)$ with a train of impulses to obtain the sampled function $f_s(t)$:

$$f_s(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad T < \frac{1}{2W}. \quad (12.50)$$

To obtain the Fourier transform of the sampled function, we use the convolution theorem:

$$\mathcal{F} \left[ f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \right] = \mathcal{F} [f(t)] \otimes \mathcal{F} \left[ \sum_{n=-\infty}^{\infty} \delta(t - nT) \right]. \quad (12.51)$$

Let us denote the Fourier transform of $f(t)$ by $F(\omega)$. The Fourier transform of a train of impulses in the time domain is a train of impulses in the frequency domain (Problem 5):

$$\mathcal{F} \left[ \sum_{n=-\infty}^{\infty} \delta(t - nT) \right] = \sigma_0 \sum_{n=-\infty}^{\infty} \delta(w - n\sigma_0) \quad \sigma_0 = \frac{2\pi}{T}. \quad (12.52)$$
Thus, the Fourier transform of $f_s(t)$ is

$$F_s(\omega) = F(\omega) \otimes \sum_{n=-\infty}^{\infty} \delta(\omega - n\sigma_0)$$

(12.53)

$$= \sum_{n=-\infty}^{\infty} F(\omega) \otimes \delta(\omega - n\sigma_0)$$

(12.54)

$$= \sum_{n=-\infty}^{\infty} F(\omega - n\sigma_0)$$

(12.55)

where the last equality is due to the sifting property of the delta function.

**Figure 12.10** Fourier transform of the sampled function.
Ideal Sampling – Time Domain View

**Figure 12.11** Effect of sampling at a rate less than 2W samples per second.

**Figure 12.12** Aliased reconstruction.
Discrete Fourier Transform (DFT)

- So far, we considered continuous functions
- What about discrete signal samples?
- Recall the Fourier series of periodic function \( f(t) \) with period \( T \)
  \[
  c_k = \frac{1}{T} \int_{0}^{T} f(t) e^{i k \omega_0 t} dt
  \]
  Assume we sample the function \( N \) time during each \( T \) period.
  \[
  F_k = \frac{1}{T} \int_{0}^{T} f(t) \sum_{n=0}^{N-1} \delta \left( t - \frac{n}{N} T \right) e^{i k \omega_0 t} dt = \frac{1}{T} \sum_{n=0}^{N-1} f \left( \frac{n}{N} T \right) e^{\frac{2\pi k n}{N}}
  \]
Taking $T=1$ and Defining

Then the periodic coefficients

$$F_k = \sum_{n=0}^{N-1} f_n e^{\frac{i2\pi kn}{N}}$$

Since

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_0 kt}$$

for $t = \frac{n}{N} T$:

$$f_n = f\left(\frac{n}{N} T\right) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{i2\pi kn}{N}}$$
$$f_n = \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} c_{k+lN} e^{i \frac{2\pi n (k+lN)}{N}}$$

Note: $e^{i \frac{2\pi n (k+lN)}{N}} = e^{i \frac{2\pi nk}{N}} e^{i 2\pi nl} = e^{i \frac{2\pi nl}{N}}$

$$f_n = \sum_{k=0}^{N-1} e^{i 2\pi nk} \sum_{l=-\infty}^{\infty} c_{k+lN}$$

$$\bar{c}_k \equiv \sum_{l=-\infty}^{\infty} c_{k+lN}$$

$\bar{c}_k$ is periodic with period $N$, $\bar{c}_k = F_k / N$

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{i \frac{2\pi kn}{N}}$$
References and Homework

- References
    - http://www.csie.nctu.edu.tw/~cmliu/Courses/dsp/
  - Chapter 5, Steven J. Leon, “Linear Algebra with Applications,” Seventh Edition

- Homeworks
  - 2, 3, 4