1. Show that the family of regular languages is closed under finite union and intersection, that is, if $L_1, L_2, \ldots, L_n$ are regular, then

$$L_U = \bigcup_{i=1,2,\ldots,n} L_i \quad \text{and} \quad L_I = \bigcap_{i=1,2,\ldots,n} L_i$$

are also regular.

**Ans.** The following shows that $L_U$ and $L_I$ are regular if $L_1, L_2, \ldots, L_n$ are regular.

- For $L_U$, if $L_1, L_2, \ldots, L_n$ are regular, there exists regular expressions $r_1, r_2, \ldots, r_n$ such that $L_1 = L(r_1), L_2 = L(r_2), \ldots, L_n = L(r_n)$. By definition, $r_1 + r_2 + \ldots + r_n$ is the regular expression denoting $L_1 \cup L_2 \cup \ldots \cup L_n$. Thus, closure under finite union is immediate.

- For $L_I$, let $L_1 = L(M_1), L_2 = L(M_2), \ldots, L_n = L(M_n)$, where $M_1 = (Q_1, \Sigma, \delta_1, q_{1,0}, F_1)$, $M_2 = (Q_2, \Sigma, \delta_2, q_{2,0}, F_2), \ldots, M_n = (Q_n, \Sigma, \delta_n, q_{n,0}, F_n)$ are DFAs. We construct from $M_1, M_2, \ldots, M_n$ a combined automaton $M = (Q, \Sigma, \delta, q_0 = (q_{1,0}, q_{2,0}, \ldots, q_{n,0}), F)$, whose state set $Q = Q_1 \times Q_2 \times \ldots \times Q_n$ consists of tuples $(q_{1,i_1}, q_{2,i_2}, \ldots, q_{n,i_n})$ whenever $M_k$ is in state $q_{k,i_k}$, $1 \leq k \leq n$. This is achieved by taking $\delta((q_{1,i_1}, q_{2,i_2}, \ldots, q_{n,i_n}), a) = (q_{1,j_1}, q_{2,j_2}, \ldots, q_{n,j_n})$, whenever $\delta_1(q_{1,i_1}, a) = q_{1,j_1}, \delta_2(q_{2,i_2}, a) = q_{2,j_2}, \ldots, \delta_n(q_{n,i_n}, a) = q_{n,j_n}$ for all $a \in \Sigma$. $F$ is defined as the set of all $(q_{1,i_1}, q_{2,i_2}, \ldots, q_{n,i_n})$ such that $q_{1,i_1} \in F_1, q_{2,i_2} \in F_2, \ldots,$ and $q_{n,i_n} \in F_n$. Now we have that $w \in L_1 \cap L_2 \cap \ldots \cap L_n$ if and only if it is accepted by $M$. Consequently, $L_I$ is regular.

2. Define the complementary (or cor) of two languages by

$$\text{cor}(L_1, L_2) = \{ w : w \in \overline{T_1} \text{ or } w \in \overline{T_2} \}.$$ 

Show that the family of regular languages is closed under the cor operation.

**Ans.** $\text{cor}(L_1, L_2) = \{ w : w \in \overline{T_1} \text{ or } w \in \overline{T_2} \} = \{ w : w \in \overline{T_1} \} \cup \{ w : w \in \overline{T_2} \}$. If $L_1$ and $L_2$ are regular, let $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ be DFAs that accept $L_1$ and $L_1$, respectively. Then the DFAs $\overline{M}_1 = (Q_1, \Sigma, \delta_1, q_{01}, Q_1 - F_1)$ and $\overline{M}_2 = (Q_2, \Sigma, \delta_2, p_{02}, Q_2 - F_2)$ accept $\overline{T_1}$ and $\overline{T_2}$. Thus, $\{ w : w \in \overline{T_1} \}$ and $\{ w : w \in \overline{T_2} \}$ are regular. Then by the results of the last problem, we know that the family of regular languages is closed under finite union. Therefore, we conclude that $\text{cor}(L_1, L_2)$ is regular.

3. Let $L_1 = L(a^*bba^*)$ and $L_2 = L(aba^*)$. Find $L_1/L_2$.

**Ans.** We first construct a DFA that accepts $L_1$ as follows. We check each state $q_0, q_1, q_2,$ and $q_3$ to see whether there is a walk labeled $aba^*$ to the final state $q_2$. We see that only $q_0$ qualifies. Thus, the result $L_1/L_2 = L(a^*)$ clearly.
4. The **left quotient** of a language $L_1$ with respect to $L_2$ is defined as

$$L_2/L_1 = \{ y : x \in L_2, xy \in L_1 \}.$$ 

Show that the family of regular languages is closed under the left quotient with a regular language.

**Ans.** We have that the reverse language of $L_2/L_1$ is

$$(L_2/L_1)^R = \{ y^R : y^R x^R \in L_1^R, x^R \in L_2^R \} = L_1^R/L_2^R.$$ 

If $L_1$ and $L_2$ are regular, from the result of homework 1-(12), we have that $L_1^R$ and $L_2^R$ are regular. Now, because the right-quotient of $L_1^R$ with $L_2^R$, i.e., $L_1^R/L_2^R$, is regular (from Theorem 4.4 in the textbook), we have that $(L_2/L_1)^R$ is regular. Again, from the result of homework 1-(12), we have that $(L_2/L_1)^R = L_2/L_1$ is regular.

5. Show that there exists an algorithm for determining if $\lambda \in L$, for any regular language $L$.

**Ans.** We represent the language $L$ by some DFA, then test $\lambda$ to see if it is accepted by this automaton.

6. Exhibit an algorithm for determining whether or not a regular language $L$ contains any string $w$ such that $w^R \in L$.

**Ans.** The language that contains any string $w$ such that $w^R \in L$ can be defined as $L \cap L^R$. By closure, we know that $L \cap L^R$ is regular. Thus we represent the language $L \cap L^R$ by some DFA, then test $w$ to see if it is accepted by this automaton.

7. Prove that $L = \{ a^n b^k : k \neq n + 1 \}$ is not regular.

**Ans.** Assume that $L$ is regular, so that the pumping lemma must hold. Given some positive integer $m$, we pick a string $w = a^m b^m a^{2m+1} \in L$. Because of the constraint $|xy| \leq m$ and $|y| \geq 1$, $y$ must be all $b$'s, that is, $y = b^z, 1 \leq z \leq m$. We now pump up, using $i = 2$. The result string $w_2 = a^{m+z} b^m a^{2m+1}$ is not in $L$ when $z = 1$. This contradicts the pumping lemma and thereby indicates that the assumption that $L$ is regular must be false.

8. Prove that $L = \{ w w : w \in \{ a, b \}^* \}$ is not regular.

**Ans.** Assume that $L$ is regular, so that the pumping lemma must hold. Given some positive integer $m$, we pick a string $ww \in L$ with $|w| = m$. Because of the constraint $|xy| \leq m$ and $|y| \geq 1$, $y$ must be a substring of $w$ in the tail. We now pump up, using $i = 0$. The result string is not in $L$ clearly. This contradicts the pumping lemma and thereby indicates that the assumption that $L$ is regular must be false.

9. For $\Sigma = \{ a \}$, determine whether or not $L = \{ a^n : n = k^3 \text{ for some } k \geq 0 \}$ is regular.

**Ans.** Assume that $L$ is regular, so that the pumping lemma must hold. Given some positive integer $m$, we pick a string $w = a^m \in L$. Because of the constraint $|xy| \leq m$ and $|y| \geq 1$, $y$ must be all $a$’s, that is, $y = a^z, 1 \leq z \leq m$. We now pump up, using $i = 2$. The result string
$w_0 = a^{m^3 + z}$ is not in $L$ because $m^3 < m^3 + z < (m + 1)^3$. This contradicts the pumping lemma and thereby indicates that the assumption that $L$ is regular must be false.

10. For each of the following languages, make a conjecture whether to not it is regular. Then prove your conjecture.

(a) $L = \{a^nb^l : n/l \text{ is an integer}\}$.
(b) $L = \{a^nb^l : n \leq l \leq 2n\}$.
(c) $L = \{a^nb^l : n > 100, l \leq 100\}$.

Ans. The language in (a) and (b) are not regular and the language in (c) is regular, we prove our conjecture as follows.

(a) Assume that $L$ is regular, so that the pumping lemma must hold. Given some positive integer $m$, we pick a string $w = a^mb^m \in L$. Because of the constraint $|xy| \leq m$ and $|y| \geq 1$, $y$ must be all $a$’s, that is, $y = a^z$, $1 \leq z \leq m$. We now pump up, using $i = 0$. The result string $w_0 = a^{m-z}b^m$ is not in $L$ when $z = 1, 2, \ldots, m - 1$. This contradicts the pumping lemma and thereby indicates that the assumption that $L$ is regular must be false.

(b) Assume that $L$ is regular, so that the pumping lemma must hold. Given some positive integer $m$, we pick a string $w = a^mb^m \in L$. Because of the constraint $|xy| \leq m$ and $|y| \geq 1$, $y$ must be all $a$’s, that is, $y = a^z$, $1 \leq z \leq m$. We now pump up, using $i = 2$. The result string $w_2 = a^{m+z}b^m$ is not in $L$ when $z = 1, 2, \ldots, m$. This contradicts the pumping lemma and thereby indicates that the assumption that $L$ is regular must be false.

(c) The regular expression of $L$ is $a^{101}a^*(\lambda + b + b^2 + \ldots + b^{100})$. 
