Discrete Mathematics (2009 Spring)
Counting (Chapter 5, 4 hours)

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Combinatorics

- The study of the number of ways to put things together into various combinations.
- *E.g.* In a contest entered by 100 people, how many different top-10 outcomes could occur?
- *E.g.* If a password is 6-8 letters and/or digits, how many passwords can there be?

Fundamental Tools

- Sum and product rules
- The inclusion-exclusion principle
- Tree diagrams
- The pigeonhole principle

§5.1 The Basics of Counting

Sum and Product Rules

- Let m be the number of ways to do task 1 and n the number of ways to do task 2 (with each number independent of how the other task is done), and assume that no way to do task 1 simultaneously also accomplishes task 2.
- The *sum rule*: The task "do either task 1 or task 2, but not both" can be done in $m + n$ ways.
- The *product rule*: The task "do both task 1 and task 2" can be done in mn ways.
- What is the diagram representation for each case?

Examples of the Product Rule

- The chairs of an auditorium are to be label with a letter and a positive integer less than 100. What is the largest number of chairs that can be labeled differently?
  - Ans: $26 * 99 = 2574$.
- How many different bit strings are there of length seven?
  - Ans: $2 * 2 * 2 * 2 * 2 * 2 * 2 = 2^7$.
- How many functions are there from a set with m elements to one with n element?
  - Ans: n^m.
- How many one-to-one functions are there from a set with m elements to one with n element? $(n \geq m)$
  - Ans: $n(n-1)(n-2)\ldots(n-m+1)$.

An Examples of the Product Rule

Example

What is the value of k after the following code has been executed?

$k = 0$
for $i_1 = 1$ **to** n_1
  for $i_2 = 1$ **to** n_2
    ⋮
      for $i_m = 1$ to n_m
        $k = k + 1$

Solution

$n_1 \cdot n_2 \cdot \ldots \cdot n_m$

Examples of the Sum Rule

- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. How many possible projects are there to choose from?

  Ans $23 + 15 + 19 = 57$.

- Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

  Ans $P = P6 + P7 + P8$.
  $P6 = 36^6 - 26^6 = 1,867,866,560$.
  $P7 = 36^7 - 26^7 = 70,332,353,920$.
  $P8 = 36^8 - 26^8 = 2,612,282,842,880$.
  $P = 2,684,483,063,360$.

The Number of IP Addresses

- Some facts about IPv4:
  - Valid computer addresses are in one of 3 types:
    - A class A IP address contains a 7-bit "netid" $\neq 1^7$, and a 24-bit "hostid"
    - A class B address has a 14-bit netid and a 16-bit hostid.
    - A class C addr. has 21-bit netid and an 8-bit hostid.
  - The 3 classes have distinct headers $(0, 10, 110)$
  - Hostids that are all 0s or all 1s are not allowed.
- How many valid computer addresses are there?

IP address solution

- (# addrs) = (# class A) + (# class B) + (# class C)
- (# class A) = (# valid netids) $\cdot$ (# valid hostids)
- (# valid class A netids) $= 2^7 - 1 = 127$.
- (# valid class A hostids) $= 2^{24} - 2 = 16,777,214$.
- Continuing in this fashion we find the answer is:
  - 3,737,091,842 (3.7 billion IP addresses)

The Inclusion-Exclusion Principle

- Let m be the number of ways to do task 1 and n the number of ways to do task 2 (with each number independent of how the other task is done), and suppose that $k > m, n$ of the ways of doing task 1 also simultaneously accomplish task 2.
- Then the number of ways to accomplish "Do either task 1 or task 2" is $m + n - k$.
- Set theory: If A and B are not disjoint, then $|A \cup B| = |A| + |B| - |A \cap B|$.
- General Formula

$$\left|\bigcup_{1\le i\le n} A_i\right| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \ldots..$$

An Inclusion-Exclusion Example

- Hypothetical rules for passwords:
  - Passwords must be 2 characters long.
  - Each password must be a letter $a - z$, a digit $0 - 9$, or one of the 10 punctuation characters !@#$%^& * ().
  - Each password must contain at least 1 digit or punctuation character.
- A legal password has a digit or punctuation character in position 1 *or* position 2. These cases overlap, so the principle applies.
  - (# of passwords w. OK symbol in position 1) = $(10 + 10) \cdot (10 + 10 + 26)$
  - (# of passwords w. OK symbol in position 2) = $20 \cdot 46$
  - (# of passwords w. OK symbol in both places) = $20 \cdot 20$

Ans: $920 + 920 - 400 = 1,440$

Tree Diagrams

- How many bit strings of length four do not have two consecutive 1's?

§5.2 The Pigeonhole Principle

Pigeonhole Principle

- A.k.a. Dirichlet drawer principle
- If $\geq k+1$ objects are assigned to k places, then at least 1 place must be assigned ≥ 2 objects.
- In terms of the assignment function:
  - If $f : A \rightarrow B$ and $|A| \geq |B| + 1$, then some element of B has ≥ 2 preimages under f.
  - I.e., f is not one-to-one.

Examples of the Pigeonhole Principle

- There are 101 possible numeric grades ($0 - 100$) rounded to the nearest integer.
- There are > 101 students in this class.
- Therefore, there must be at least one (rounded) grade that will be shared by at least 2 students at the end of the semester.
  - I.e., the function from students to rounded grades is *not* a one-to-one function.

Fun Pigeonhole Proof (Ex. 4, p.314)

Theorem

$\forall n \in N$, $\exists$ a multiple $m > 0$ of n s.t. m has only 0's and 1's in its decimal expansion!

Proof.

Consider the $n+1$ decimal integers $1, 11, 111, \cdots, 11\cdots1$. They have only n possible residues mod n. So, take the difference of two that have the same residue. The result is the answer! □

A Specific Example

- Let $n = 3$. Consider $1, 11, 111, 1111$.
  - $1 \bmod 3 = 1$
  - $11 \bmod 3 = 2$
  - $111 \bmod 3 = 0$ $\Leftarrow$Lucky extra solution.
  - $1,111 \bmod 3 = 1$
- $1,111 - 1 = 1,110 = 3 \cdot 370$.
  - It has only 0's and 1's in its expansion.
  - Its residue $\bmod 3 = 0$, so it's a multiple of 3.

Generalized Pigeonhole Principle

- If N objects are assigned to k places, then at least one place must be assigned at least $\lceil N/k \rceil$ objects.
- E.g., there are $N = 280$ students in this class. There are $k = 52$ weeks in the year.
  - Therefore, there must be at least 1 week during which at least $\lceil 280/52 \rceil = \lceil 5.38 \rceil = 6$ students in the class have a birthday.

Proof of G.P.P.

- By contradiction. Suppose every place has $< \lceil N/k \rceil$ objects, thus $\leq \lceil N/k \rceil - 1$.
- Then the total number of objects is at most

$$
\begin{aligned}
k\left(\lceil N/k \rceil - 1\right) &< k\left(\left(N/k + 1\right) - 1\right) \\
&= k\left(N/k\right) = N
\end{aligned}
$$

- So, there are less than N objects, which contradicts our assumption of N objects!

G.P.P. Example

- Given: There are 280 students in the class. Without knowing anybody's birthday, what is the largest value of n for which we can prove that at least n students must have been born in the same month?

Ans $\lceil 280/12 \rceil = \lceil 23.3 \rceil = 24$

§5.3 Permutations and Combinations
§5.5 Generalized Permutations and Combinations

Permutations

- A permutation of a set S of objects is a sequence containing each object once.
- An ordered arrangement of r distinct elements of S is called an r-permutation.

Theorem

The number of r-permutations of a set with $n = |S|$ elements is

$$P(n, r) = n(n-1)\dots(n-r+1) = \frac{n!}{(n-r)!}.$$

Permutation Examples

- A terrorist has planted an armed nuclear bomb in your city, and it is your job to disable it by cutting wires to the trigger device. There are 10 wires to the device. If you cut exactly the right three wires, in exactly the right order, you will disable the bomb, otherwise it will explode! If the wires all look the same, what are your chances of survival?
- How many permutations of the letters ABCDEFGH contain the string ABC?

Permutations with Repetition

Theorem

The number of r-permutations of a set of n objects with repetition allowed is

$$n^r.$$

Example

If $|A| = r$ and $|B| = n$, how many functions are there from A to B?

Permutations with Indistinguishable Objects

Theorem

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Example

How many different strings can be made by reordering the letters of the word "SUCCESS"?

Combinations

- An r-combination of elements of a set S is simply a subset $T \subseteq S$ with r members, $|T| = r$.
- The number of r-combinations of a set with $n = |S|$ elements is denoted by $C(n, r)$.
  - We have $C(n, r) = C(n, n - r)$ since choosing the r members of T is the same thing as choosing the $n - r$ non-members of T.

Theorem

The number of r-combinations of a set with $n = |S|$ elements is

$$\begin{aligned} C(n, r) &= \binom{n}{r} = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!} \\ &= \frac{n!}{r!(n-r)!} \end{aligned}$$

Example

How many distinct 7-card hands can be drawn from a standard 52-card deck?

Solution

The order of cards in a hand doesn't matter. So,

$$\begin{aligned} C(52,7) &= P(52,7)\,/\,P(7,7) \\ &= 52\cdot 51\cdot 50\cdot 49\cdot 48\cdot 47\cdot 46/7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 \\ &= 52\cdot 17\cdot 10\cdot 7\cdot 48\cdot 46 = 133,784,560 \end{aligned}$$

Example

How many bit strings of length n contain exactly r 1's?

Combination with Repetition

Example

How many ways are there to select four pieces of fruits from a bowl containing apples, oranges and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl.

Solution

Enumerated by tree diagrams. (Can you figure out another solutions?)

Combination with Repetition (Cont.)

Theorem (Combinations with repetition)

There are $C(n+r-1, r)$ r-combinations from a set with n elements when repetition of elements is allowed.

Proof.

*Consider the permutation of r *'s (stars) divided by $n-1$ $|$'s (bars).* □

Example

How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, and x_3 are nonnegative integers?

Distributing Objects into Boxes

Theorem

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, for $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Example

How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

§5.4 Binomial Coefficients

Binomial Coefficients

Theorem (The Binomial Theorem)

Let x and y be variables, and let n be a nonnegative integer. Then,

$$\begin{aligned}(x+y)^n &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 \\ &\quad + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n\end{aligned}$$

Examples

Let n be a nonnegative integer. Then

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} &= 2^n, \\ \sum_{k=0}^{n} (-1)^k \binom{n}{k} &= 0, \\ \sum_{k=0}^{n} 2^k \binom{n}{k} &= 3^n. \end{aligned}$$

Some Identities

Theorem (Pascal's Identity)

Let n and k be positive integers with $n \geq k$. Then,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof.

Two proof approaches: (a) algebraic proof; (b) combinatorial proof. □

Some Identities (Cont.)

Theorem (Vandermonde's Identity)

Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Theorem

Let n and r be nonnegative integers with $r \leq n$. Then,

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

§5.6 Generating Permutations and Combinations

Generating Permutations

- Lexicographic ordering
  - Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be permutations of $\{1, 2, \cdots, n\}$. We say $a_1, a_2, \cdots, a_n$ precedes $b_1, b_2, \cdots, b_n$ if for some k, with $1 \le k \le n$, $a_1 = b_1$, $a_2 = b_2$, $\cdots$, $a_{k-1} = b_{k-1}$, and $a_k < b_k$.
  - For example: $15324 < 15432$ and $cadebf < cafdeb$.
- How to generate all permutations?
  - Start from the smallest one, and then keep finding the next largest one.

Find the Next Largest Permutation

Example

Find the next largest permutation in lexicographic order of 6725431.

1. Search from the tail to find the first decreasing element. Let's say it is at position k and call the subsequence after this is the tail. For example, '2' is the one at position 3 place and "5431" is the tail.
2. Switch the element at position k with the smallest element in the tail that is larger than the element at the position k. For example, '2' and '3' are switched, and we have "6735421".
3. Reverse the tail. For example, "5421" is reverse, and we have "6731245".

Pseudocode for Finding the Next Largest Permutation

procedure next_permutation($a_1, a_2, \cdots, a_n$: permutation of $\{1, 2, \ldots, n\}$ not equal to $n, n-1, \cdots, 2, 1$)
$j = n - 1$;
while $(a_j > a_{j+1})$ $j{=}j-1$;
$k = n$;
while $(a_j > a_k)$ $k = k - 1$;
Interchange a_j and a_k;
$r = n$; $s = j + 1$;
while $(r > s)$
begin
  interchange a_r and a_s;
  $r = r - 1$; $s = s + 1$;
end

Examples

1. Find the next largest permutation in lexicographic order of 462531.
2. Find the next three largest permutation in lexicographic order of *badecf*.

Problem

Find all permutations of an n-element set.

1. *Begin with the smallest permutation.*
2. *Find the next largest permutation until we have the largest permutation.*
   - *How can we know when we have the largest one?*

Generating Combinations

- Selection v.s.$\{0,1\}$-string
  - S is a set of n elements, and elements of S are with ordering.
  - A subset of S can be represented by a string of $\{0,1\}^n$
- For example, $S = \{1,2,3,4,5\}$. Then,
  - $\{1,3\}$ is corresponding to 10100.
  - $\{1,3,4,5\}$ is corresponding to 10111.
- Find all combinations of an n-element set.
  - Go through all strings in $\{0,1\}^n$ by adding 1 from 0 to $2^n - 1$.
  - List corresponding combinations.

Pseudocode for Finding the Next Largest Bit String

procedure next_bit_permutation($b_{n-1}b_{n-2}\ldots b_2b_1$: bit string not all 1's)
$i = 0$;
while $(b_i == 1)$
begin
    $b_i = 0$;
    $i = i + 1$;
end
$b_i = 1$;

Generating r-Combinations

- Define the ordering of r-combinations.
  - For example,
- How to generate all r-combinations?
  - Start from the smallest one.
  - Keep finding the next largest one until we have the largest r-combination.
    - How can we know when we have the largest one?

The Next Largest r-Combination

Problem

Find the next largest r-combinations after $a_1, a_2, \cdots, a_r$ *of an n-element set.*

Solution

1. *Locate the last element* a_i *in the sequence such that* $a_i \neq n - r + i$.
2. *Replace* a_i *with* a_{i+1} *and* a_j *with* $a_i + 1 + (j - i)$ *for* $j = i + 1, i + 2, \ldots, r$.

Example

What is the next largest 4-combination of $\{1, 2, 3, 4, 5, 6\}$ after $\{1, 2, 5, 6\}$.

1. Here $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, and $a_4 = 6$.
2. We have $i = 2$. Then, $a_2 = 3$, $a_3 = 4$, and $a_4 = 5$.
3. $\{1, 3, 4, 5\}$ is the next one after $\{1, 2, 5, 6\}$.

Pseudocode for Generating the Next r-Combination

procedure next r-permutation($\{a_1, a_2, \ldots, a_r\}$: proper subset of $\{1, 2, \ldots, n\}$ not equal to $\{n - r + 1, \ldots, n\}$ with $a_1 < a_2 < \ldots < a_r$)
$i = r$;
while $(a_i == n - r + i)$ $\quad i = i - 1$;
$a_i = a_i + 1$;
for $j = i + 1$ to r
$\quad a_j = a_i + j - i$