1.15. Let $N_1$ be this NFA:

![Diagram of NFA]

Then follow the construction of this exercise, we get $N$:

![Diagram of NFA]

It is easy to check that $L(N_1) = 11(111)^*$ (denote as $A_1$) and $111 \notin A_1^*$. But $111 \in L(N)$.

1.30. In fact, $s$ can be pumped. Let $x = \epsilon, y = 0^p$ and $z = 1^p$. It is easy to check $s = xyz$ and for every integer $i \geq 0$, $xy^iz \in 0^*1^*$. (Note: the “easy” part should be written in detail if you would like to have 10pts.)

1.31. Since $A$ is regular, there exists a DFA $D = (Q, \Sigma, \delta, q, F)$ recognizing $A$. We prove $A^R$ is regular by constructing an $\epsilon$-NFA $N$ recognizing it. Define $N = (Q \cup \{q'\}, \Sigma, \delta', q', \{q\})$ where $\delta'(q', \epsilon) = F$ and for $x \in Q$ and $a \in \Sigma$, $\delta'(x, a) = \{ y : \delta(y, a) = x \}$. $\delta'(\cdot, \cdot) = \phi$ on all the other inputs. We verify that $N$ recognizes $A^R$ by prove the following two facts.

- $N$ accepts every string $\omega \in A^R$
- Every string $\omega'$ accepted by $N$ is in $A^R$.

For any string $\omega = o_1 \ldots o_\ell \in A$ of length $\ell$ and $i > 0$, let $q_0 = q$ and $q_i = \delta(q_{i-1}, o_i)$. Since $\omega$ can be accepted by $D$, we have $q_\ell \in F$. It is easy to check $q', q_\ell, q_{\ell-1}, \ldots, q_1, q_0$ is an accepting path in $N$. $\omega^R$ must be accepted by $N$, thus we conclude that $N$ accepts every string in $A^R$.

For any string $\omega'$ accepted by $N$, there exists an accepting path $q', q_1, \ldots, q_m$ in $N$ corresponding to $\omega'$ where $q_1 \in F$ and $q_m = q$. It is easy to check that $q_m, q_{m-1}, \ldots, q_1$ is also an accepting path in $D$ and the corresponding string is exactly $\omega'^R$.

1.32. Construct a 3 state NFA $N = (\{s, y, n\}, \Sigma_3, \delta, s, \{y\})$ where $\delta$ is defined as
The rest is to show that $N$ recognizes $B^R$. Let $A^R = \{ \omega : \omega \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in B^R \}$ and $C^R = \Sigma^* - (B^R \cup A^R)$. Note that in any string of length $n$ in $A$, the difference between the bottom row and the sum of top two row is $2^n$. Similarly, in any string of length $n$ in $C$, the difference between the bottom row and the sum of top two row is not divisible $2^n$.

First, we claim string $\omega$ of length $n > 0$, we have

$$\delta^*(s, \omega) = \begin{cases} \{n\}, & \omega \in A^R \\ \{y\}, & \omega \in B^R \\ \{\}, & \omega \in C^R \end{cases}$$

We prove this fact by induction on the string length. To check the induction basis, $n = 1$, is easy. Assume this claim is true for $n < k$. Consider $n = k$ and $\omega = \omega' a$ where $a \in \Sigma_3$. If $\omega' \in C^R$, then $\omega$ is also in $C^R$ since the lower digits do not have difference divisible by $2^k$. If $\omega' \in B^R$ or $\omega' \in A^R$, then we know the claim holds by carefully checking the transition table. (Note: the “carefully” part consists of several cases but not hard to complete.)

At last, observe that $\epsilon$ would be trapped in $s$. We can conclude that $N$ recognizes $B^R$, hence $B$ is regular.

1.46.c. Suppose $L$ is regular, then by the fact $\Sigma^*$ is regular we have $\bar{L} := \Sigma^* \setminus L$ must be regular. Let $p$ be the pumping length for $\bar{L}$. Observe that $1^n01^n \in \bar{L}$ $\forall n \in \mathbb{Z}^+$, so $s := 1^p01^p \in \bar{L}$. Then the pumping lemma implies that $s = xyz$ with $|y| > 0$ and $|xy| \leq p$, which means $y = 1^k$ where $1 \leq k \leq p$. The pumping lemma also implies $\forall i \geq 0$, $xy^iz \in \bar{L}$, hence, $1^{p-k}01^p \in \bar{L}$. But $p - k \leq p - 1$, which is a contradiction! So $L$ can’t be regular.