Chapter 3 Fourier Representations of Signals and Linear Time-Invariant Systems

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  - Discrete-Time Nonperiodic Signals
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Chapter 3 Fourier Representations of Signals and Linear Time-Invariant Systems

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Introduction

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2. Complex Sinusoids and Frequency Response of LTI Systems.

- The input-output relation of an LTI system is characterized by convolution with the system's impulse response.

- By representing the input as a sum of complex sinusoids, we obtain a more straightforward understanding of the input-output relation of an LTI system.
2. Complex Sinusoids and Frequency Response of LTI Systems

- **Discrete-Time LTI System**
  - The output of a complex sinusoidal input to an LTI system is a complex sinusoid of the same frequency as the input, multiplied by the frequency response of the system.

\[
y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] = \sum_{k=-\infty}^{\infty} h[k] e^{j\Omega(n-k)}
\]

\[
y[n] = e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k} = H(e^{j\Omega}) e^{j\Omega n}
\]

**Frequency Response**

\[
H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega}
\]
2. Complex Sinusoids and Frequency Response of LTI Systems

Continuous-Time LTI System

\[ y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)} d\tau = e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega \tau} d\tau = H(j\omega)e^{j\omega t} \]

Frequency response:

\[ H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega \tau} d\tau \]

Polar form complex number \( c = a + jb \):

\[ c = |c|e^{j\arg\{c\}} \text{ where } |c| = \sqrt{a^2 + b^2} \text{ and } \arg\{c\} = \tan^{-1}\left(\frac{b}{a}\right) \]

Polar form for \( H(j\omega) \):

\[ H(j\omega) = |H(j\omega)|e^{j\arg\{H(j\omega)\}} \]

where \( |H(j\omega)| = \text{Magnitude response} \) and \( \arg\{H(j\omega)\} = \text{Phase response} \)
2. Complex Sinusoids and Frequency Response of LTI Systems

**RC Circuit: Frequency response**

Frequency response:

\[
H(j\omega) = \frac{1}{RC} \int_{-\infty}^{\infty} e^{-\frac{\tau}{RC}} u(\tau)e^{-j\omega \tau} d\tau = \frac{1}{RC} \int_{0}^{\infty} e^{-\left(j\omega + \frac{1}{RC}\right)\tau} d\tau
\]

\[
= \frac{1}{RC} \left. \frac{-1}{j\omega + \frac{1}{RC}} e^{-\left(j\omega + \frac{1}{RC}\right)\tau} \right|_0^\infty
\]

\[
= \frac{1}{RC} \left( j\omega + \frac{1}{RC} \right)(0 - 1)
\]

\[
= \frac{1}{RC} \left( j\omega + \frac{1}{RC} \right)
\]

\[
= \frac{1}{RC} \left( j\omega + \frac{1}{RC} \right)
\]

\[
|H(j\omega)| = \frac{1}{RC \sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}}
\]

Phase response:

\[
\arg\{H(j\omega)\} = -\arctan(\omega RC)
\]
2. Complex Sinusoids and Frequency Response of LTI Systems

| \( |H(j\omega)| \) |
|---|
| \( \frac{1}{\sqrt{2}} \) |
| \( 1 \) |
| \( \frac{1}{RC} \) |
| \( 0 \) |
| \( \frac{1}{RC} \) |

<table>
<thead>
<tr>
<th>( \arg{H(j\omega)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
</tr>
<tr>
<td>( 0 )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
</tr>
</tbody>
</table>

\[ \psi(t) \xrightarrow{H} \lambda \psi(t) \xrightarrow{e^{j\omega t}} H(j\omega) e^{j\omega t} \xrightarrow{e^{j\Omega n}} H(e^{j\Omega}) e^{j\Omega n} \]
2. Complex Sinusoids and Frequency Response of LTI Systems

- **Eigenvalues and Eigenfunctions of a LTI System**

  If $e_k$ is an eigenvector of a matrix $A$ with eigenvalue $\lambda_k$, then

  $$Ae_k = \lambda_k e_k$$

  **Eigenfunction**
  
  $$\psi(t) = e^{j\omega t}$$
  
  $$\psi[n] = e^{j\Omega n}$$

  **Eigenvalue**
  
  $$\lambda = H(j\omega)$$
  
  $$\lambda = H(e^{j\Omega})$$

  **Arbitrary input = weighted superpositions of eigenfunctions**

  **Convolution operation** $\Rightarrow$ **Multiplication**

  **Ex. Input:**
  
  $$x(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t}$$
  
  $$x[n] = \sum_{k=1}^{M} a_k e^{j\Omega_k n}$$

  **Output:**
  
  $$y(t) = \sum_{k=1}^{M} a_k H(j\omega)e^{j\omega_k t}$$
  
  $$y[n] = \sum_{k=1}^{M} a_k H(e^{j\Omega})e^{j\Omega_k n}$$
3. Fourier Representations for Four Classes of Signals

**Periodic Signals: Fourier Series Representations**

1. \( x[n] = \text{discrete-time signal with fundamental period} N \). DTFS of \( x[n] \) is

\[
x[n] = \sum_{k} A[k] e^{j\Omega_0 n}
\]

(3.4) \( \Omega_o = \frac{2\pi}{N} \equiv \text{Fundamental frequency of} \ x[n] \)

2. \( x(t) = \text{continuous-time signal with fundamental period} T \). FS of \( x(t) \) is

\[
x(t) = \sum_{k} A[k] e^{j\omega_0 t}
\]

(3.5) \( \omega_o = \frac{2\pi}{T} \equiv \text{Fundamental frequency of} \ x(t) \)

“\(^*\)” denotes approximate value. \( A[k] = \text{the weight applied to the} \ k^{th} \text{harmonic.} \)

\( e^{j\omega_0 t} \) is the \( k^{th} \) harmonic of \( e^{j\omega_0 t} \).

<table>
<thead>
<tr>
<th>Time Property</th>
<th>Periodic</th>
<th>Nonperiodic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous ((t))</td>
<td>Fourier Series (FS)</td>
<td>Fourier Transform (FT)</td>
</tr>
<tr>
<td>Discrete ([n])</td>
<td>Discrete-Time Fourier Series (DTFS)</td>
<td>Discrete-Time Fourier Transform (DTFT)</td>
</tr>
</tbody>
</table>
3. Fourier Representations for Four Classes of Signals

3. The complex sinusoids \( \exp(jk\Omega_0 n) \) are \( N \)-periodics in the frequency index \( k \).

\[
e^{j(N+K)\Omega_0 n} = e^{jN\Omega_0 n} e^{jk\Omega_0 n} = e^{j2\pi n} e^{jN\Omega_0 n} = e^{jk\Omega_0 n}
\]

There are only \( N \) distinct complex sinusoids of the form \( \exp(jk\Omega_0 n) \) should be used in Eq. (3.4).

\[\hat{x}[n] = \sum_{k=0}^{N-1} A[k] e^{jk\Omega_0 n}\]

\(\uparrow\) Symmetries in \( x[k] \):

\[k = -(N - 1)/2 \text{ to } (N - 1)/2\]

4. Continuous-time complex sinusoid \( \exp(jk\omega_0 t) \) with distinct frequencies \( k\omega_0 \) are always distinct. FS of continuous-time periodic signal \( x(t) \) becomes

\[x(t) = \sum_{k=-\infty}^{\infty} A[k] e^{jk\omega_0 t}\]

\(\uparrow\) Mean-square error (MSE) between the signal and its series representation:

Discrete-time case:

\[MSE = \frac{1}{N} \sum_{n=0}^{N-1} \left| x[n] - \hat{x}[n] \right|^2 dt\]

Continuous-time case:

\[MSE = \frac{1}{T} \int_{0}^{T} \left| x(t) - \hat{x}(t) \right|^2 dt\]
3. Fourier Representations for Four Classes of Signals

- **Nonperiodic Signals: Fourier-Transform Representations**

1. FT of continuous-time signal:

\[
\hat{x}[n] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega 
\]

\(X(j\omega)/(2\pi)\) = the weight applied to a sinusoid of frequency \(\omega\) in the FT representation.

2. DTFT of discrete-time signal:

\[
\hat{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n} d\Omega 
\]

\(X(e^{j\Omega})/(2\pi)\) = the weight applied to the sinusoid \(e^{j\Omega n}\) in the DTFT representation.
3. Fourier Representations for Four Classes of Signals
## 3. Fourier Representations for Four Classes of Signals

### Periodic ↔ Discrete

**Table 3.2  The Four Fourier Representations**

<table>
<thead>
<tr>
<th>Time Domain</th>
<th>Periodic ((t, n))</th>
<th>Non-periodic ((t, n))</th>
<th>Frequency Domain ((k, \omega))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Continuous ((t))</strong></td>
<td><strong>Fourier Series</strong></td>
<td><strong>Fourier Transform</strong></td>
<td>Nonperiodic ((k, \omega))</td>
</tr>
<tr>
<td></td>
<td>(x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\omega_k t})</td>
<td>(x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(X[k] = \frac{1}{T} \int_{0}^{T} x(t) e^{-j\omega_k t} dt)</td>
<td>(X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x(t)) has period (T, \omega_o = 2\pi / T).</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Discrete ([n])</strong></td>
<td><strong>Discrete-Time Fourier Series</strong></td>
<td><strong>Discrete-Time Fourier Transform</strong></td>
<td><strong>Periodic ((k, \Omega))</strong></td>
</tr>
<tr>
<td></td>
<td>(x[n] = \sum_{k=0}^{N-1} X[k] e^{j\Omega_k n})</td>
<td>(x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{j\Omega_k n})</td>
<td>(X(e^{j\Omega}) = \sum_{n=\infty}^{\infty} x[n] e^{-j\Omega n})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x[n]) and (X[k]) have period (N, \Omega_o = 2\pi / N).</td>
<td>(X(e^{j\Omega})) has period (2\pi).</td>
<td></td>
</tr>
<tr>
<td><strong>Discrete ([k])</strong></td>
<td><strong>Continuous ((\omega, \Omega))</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4. Discrete-Time Periodic Signals: The Discrete-Time Fourier Series

1. DTFS pair of periodic signal $x[n]$:  

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j k \Omega_0 n}$$  

(3.10)  

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j k \Omega_0 n}$$  

(3.11)  

Fundamental period $= N$;  
Fundamental frequency $= \Omega_0 = 2\pi/N$  

Fourier coefficients; Frequency domain representation

Notation:

$$x[n] \xrightarrow{DTFS; \Omega_0} X[k]$$
4. Discrete-Time Periodic Signals: The Discrete-Time Fourier Series

Example 3.6

\[ x[n] = \begin{cases} 
1, & -M \leq n \leq M \\
0, & M < n < N - M 
\end{cases} \]

1. Period of DTFS coefficients = \( N \) \quad \Omega_o = 2\pi/N

2. It is convenient to evaluate Eq.(3.11) over the interval \( n = -M \) to \( n = N - M - 1 \).

\[ X[k] = \frac{1}{N} \sum_{n=-M}^{N-M-1} x[n] e^{-jk\Omega_0 n} \]

\[ = \frac{1}{N} \sum_{n=-M}^{M} e^{-jk\Omega_0 n} \]

3. For \( k = 0, \pm N, \pm 2N, \ldots \), we have

\[ e^{jk\Omega_o} = e^{-jk\Omega_o} = 1 \]

\[ X[k] = \frac{1}{N} \sum_{m=0}^{2M} 1 = \frac{2M + 1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots \]
4. Discrete-Time Periodic Signals: 
The Discrete-Time Fourier Series

3. For $k \neq 0, \pm N, \pm 2N, \ldots$, we may sum the geometric series in Eq. (3.15) to obtain

$$X[k] = \frac{e^{jk\Omega_0 M}}{N} \left( \frac{1 - e^{jk\Omega_0 M (2M+1)}}{1 - e^{-jk\Omega_0}} \right), \quad k \neq 0, \pm N, \pm 2N, \ldots$$

4. Substituting $\Omega_o = \frac{2\pi}{N}$, yields

$$X[k] = \frac{1}{N} \frac{\sin(k\Omega_0 (2M+1)/2)}{\sin(k\Omega_0 / 2)},$$

4. Substituting $\Omega_o = \frac{2\pi}{N}$, yields

$$X[k] = \begin{cases} 
\frac{1}{N} \frac{\sin(k\pi (2M+1)/N)}{\sin(k\pi / N)}, & k \neq 0, \pm N, \pm 2N, \ldots \\
(2M+1)/N, & k = 0, \pm N, \pm 2N, \ldots 
\end{cases}$$

The value of $X[k]$ for $k = 0, \pm N, \pm 2N, \ldots$, is obtained from the limit as $k \to 0$. 

$$\lim_{k \to 0, \pm N, \pm 2N, \ldots} \frac{1}{N} \frac{\sin(k\pi (2M+1)/N)}{\sin(k\pi / N)} = \frac{2M+1}{N}$$
L'Hôpital's rule

- In its simplest form, l'Hôpital's rule states that for functions $f$ and $g$:

The general form of l'Hôpital's rule covers many more cases. Suppose that $c$ and $L$ are extended real numbers (i.e., real numbers, positive infinity, or negative infinity). Suppose that either

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$$

or

$$\lim_{x \to c} f(x) = \pm \lim_{x \to c} g(x) = \pm \infty.$$  

And suppose that

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L.$$  

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L.$$  

The limits may also be one-sided limits.
4. Discrete-Time Periodic Signals: The Discrete-Time Fourier Series

The DTFS coefficients for the square wave shown in Fig. 3.11, assuming a period $N = 50$: (a) $M = 4$. (b) $M = 12$. 

**Figure 3.12** (p. 211)
4. Discrete-Time Periodic Signals: The Discrete-Time Fourier Series

- Symmetry property of DTFS coefficient: \( X[k] = X[-k] \).

1. DTFS of Eq. (3.10) can be written as a series involving harmonically related cosines.

2. Assume that \( N \) is even and let \( k \) range from \( -(N/2) + 1 \) to \( N/2 \). Eq. (3.10) becomes

\[
x[n] = \sum_{k=-N/2+1}^{N/2} X[k]e^{jk\Omega_0 n}
\]

3. Use \( X[m] = X[-m] \) and the identity \( N\Omega_o = 2\pi \) to obtain

\[
x[n] = X[0] + X[N/2]e^{jk\Omega_0 n/2} + \sum_{m=1}^{N/2-1} \left( X[m]e^{jk\Omega_0 n} + X[-m]e^{-jk\Omega_0 n} \right)
\]

4. Define the new set of coefficients

\[
e^{j\pi n} = \cos(\pi n)
\]

since \( \sin(\pi n) = 0 \) for integer \( n \).
4. Define the new set of coefficients

\[ B[k] = \begin{cases} 
X[k], & k = 0, \quad N/2 \\
2X[k], & k = 1, \quad 2, \ldots, \quad N/2 - 1 
\end{cases} \]

and write the DTFS for the square wave in terms of a series of harmonically related cosines as

\[ x[n] = \sum_{k=0}^{N/2} B[k] \cos(k\Omega_0 n) \]  \hspace{1cm} (3.17)

★ A similar expression may be derived for \( N \) odd.

**Example 3.7 Building a Square Wave From DTFS Coefficients**

\[ x_J[n] = \sum_{k=0}^{J} B[k] \cos(k\Omega_0 n) \]
4. Discrete-Time Periodic Signals

The Discrete-Time Fourier Series

(a) $J = 1$. (b) $J = 3$. (c) $J = 5$. (d) $J = 23$. (e) $J = 25$. 
Example Figure 3.15

Electrocardiograms for two different heartbeats and the first 60 coefficients of their magnitude spectra.

(a) Normal heartbeat.
(b) Ventricular tachycardia.
(c) Magnitude spectrum for the normal heartbeat.
(d) Magnitude spectrum for ventricular tachycardia.
1. Continuous-time periodic signals are represented by the Fourier series (FS).
2. A signal with fundamental period $T$ and fundamental frequency $\omega_0 = \frac{2\pi}{T}$, the $FS$ pair of $x(t)$ is

$$x(t) = \sum_{k=\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

(3.19)

$$X[k] = \frac{1}{T} \int_{0}^{T} x(t)e^{-jk\omega_0 t} dt$$

(3.20)

3. Notation:

$$x(t) \xrightarrow{FS,\omega_0} X[k]$$

The variable $k$ determines the frequency of the complex sinusoid associated with $X[k]$ in Eq. (3.19).
5. Continuous-Time Periodic Signals—The Fourier Series

- **Mean-square error (MSE)**

1. Suppose we define

   \[ \hat{x}(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \]

   then the MSE between \( x(t) \) and \( \hat{x}(t) \) is zero, or, mathematically,

   \[ \frac{1}{T} \int_{0}^{T} |x(t) - \hat{x}(t)|^2 \, dt = 0 \]

2. If \( x(t) \) is square integrable,

   \[ \frac{1}{T} \int_{0}^{T} |x(t)|^2 \, dt < \infty \]

3. Pointwise convergence of \( x(t) = \hat{x}(t) \) is guaranteed at all values of \( t \) except those corresponding to discontinuities if the Dirichlet conditions are satisfied:
   - \( x(t) \) is bounded.
   - \( x(t) \) has a finite number of maximum and minima in one period.
   - \( x(t) \) has a finite number of discontinuities in one period.

   If a signal \( x(t) \) satisfies the Dirichlet conditions and is not continuous, then \( \hat{x}(t) \) converges to the mipoint of the left and right limits of \( x(t) \) at each discontinuity.
5. Continuous-Time Periodic Signals—The Fourier Series

1. Fundamental frequency: $\omega_0 = \frac{2\pi}{T}$.

2. $X[k]$: by integrating over $-T/2 \leq t \leq T/2$

$$X[k] = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j\omega_0 t} dt = \frac{1}{T} \int_{-T_0}^{T_0} e^{-j\omega_0 t} dt$$

$$= \frac{-1}{Tj\omega_0} e^{-j\omega_0 t} \bigg|_{-T_0}^{T_0}, \quad k \neq 0$$

$$= \frac{2}{T\omega_0} \left( e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0} \right), \quad k \neq 0$$

$$= \frac{2\sin(k\omega_0 T_0)}{Tk\omega_0}, \quad k \neq 0$$

For $k = 0$, we have

$$X[0] = \frac{1}{T} \int_{-T_0}^{T_0} dt = \frac{2T_0}{T}$$

3. By means of L’Hôpital’s rule, we have
5. Continuous-Time Periodic Signals—The Fourier Series

\[
\lim_{k \to 0} \frac{2\sin(k\omega_0 T_0)}{Tk\omega_0} = \frac{2T_0}{T}
\]

\[
X[k] = \frac{2\sin(k\omega_0 T_0)}{Tk\omega_0}
\]

4. \(X[k]\) is real valued. Using \(\omega_0 = 2\pi/T\) gives \(X[k]\) as a function of the ratio \(T_0/T\): \[(3.23)\]

\[
X[k] = \frac{2\sin(2\pi T_0 / T)}{k2\pi}
\]

5. Fig. 3.22 (a)-(c) depict \(X[k]\), \(-50 \leq k \leq 50\), for \(T_0/T = 1/4\), \(T_0/T = 1/16\), and \(T_0/T = 1/64\), respectively.

Definition of Sinc function:

\[
sinc(u) = \frac{\sin(\pi u)}{\pi u}
\]

(3.24)

Fig. 3.23.

1) Maximum of sinc function is unity at \(u = 0\), the zero crossing occur at integer values of \(u\), and the amplitude dies off as \(1/u\).

2) The portion of this function between the zero crossings at \(u = \pm 1\) is manilolobe of the sinc function.
5. Continuous-Time Periodic Signals—The Fourier Series

The FS coefficients, \( X[k] \), \(-50 \leq k \leq 50\), for three square waves. (see Fig. 3.21)  
(a) \( T_0/T = 1/4 \).  
(b) \( T_0/T = 1/16 \).  
(c) \( T_0/T = 1/64 \).
5. Continuous-Time Periodic Signals—The Fourier Series

**Example 3.15**

\[ \hat{x}_J(t) = \sum_{k=1}^{J} B[k] \cos(k\omega_0 t) \]

- (b) \( J = 3 \)
- (c) \( J = 7 \)
- (d) \( J = 29 \)

Gibbs phenomenon: Ripple near discontinuities

Figure 3.25b-3 (p. 226) (b) \( J = 3 \). (c) \( J = 7 \). (d) \( J = 29 \).
5. Continuous-Time Periodic Signals—The Fourier Series

- **Trigonometric FS for Real Signal**

\[ x(t) = B[0] + \sum_{k=1}^{\infty} B[k] \cos(k\omega_0 t) + A[k] \sin(k\omega_0 t) \]  

(3.25)

**FS coefficients:**

\[
B[0] = \frac{1}{T} \int_{0}^{T} x(t)dt \\
B[k] = \frac{2}{T} \int_{0}^{T} x(t) \cos(k\omega_0 t)dt \\
A[k] = \frac{2}{T} \int_{0}^{T} x(t) \sin(k\omega_0 t)dt
\]

(3.26)

**Relation between exponential FS and trigonometric FS coefficients**

\[
B[k] = X[k] + X[-k] \\
A[k] = j(X[k] - X[-k])
\]

for \( k \neq 0 \)

\( B[0] = X[0] \) represents the time-averaged value of the signal.
1. The DTFT is used to represent a discrete-time nonperiodic signal as a superposition of complex sinusoids.

2. DTFT representation of time-domain signal $x[n]$:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n} d\Omega$$  \hspace{1cm} (3.31)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$  \hspace{1cm} (3.32)

3. Notation:

$$x[n] \xrightarrow{\text{DTFT}} X(e^{j\Omega})$$

![Diagram showing DTFT pair and frequency-domain representation of $x[n]$]
6. Discrete-Time Nonperiodic Signals—The Discrete-Time Fourier Transform

- **Condition for convergence of DTFT:**

  If \( x[n] \) is of infinite duration, then the sum converges only for certain classes of signals. If \( x[n] \) is absolutely summable, i.e.,

  \[
  \sum_{n=-\infty}^{\infty} |x[n]| < \infty
  \]
  
  The sum in Eq. (3.32) converges uniformly to a continuous function of \( \Omega \).

- If \( x[n] \) is not absolutely summable, but does satisfy (i.e., if \( x[n] \) has finite energy),

  \[
  \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty
  \]
  
  It can be shown that the sum in Eq. (3.32) converges in a mean-square error sense, but does not converge pointwise.
6. Discrete-Time Nonperiodic Signals—The Discrete-Time Fourier Transform

Let

\[ x[n] = \begin{cases} 1, & |n| \leq M \\ 0, & |n| > M \end{cases} \]

1. DTFT of \( x[n] \):

\[ X(e^{j\Omega}) = \sum_{n=-M}^{M} 1e^{-j\Omega n}. \]

Change of variable

\( m = n + M \)

The expression for \( X(e^{j\Omega}) \) when \( \Omega \neq 0, \pm 2\pi, \pm 4\pi, \ldots \)

\[ X(e^{j\Omega}) = e^{j\Omega M} \frac{e^{-j\Omega(2M+1)/2} \left( e^{j\Omega(2M+1)/2} - e^{-j\Omega(2M+1)/2} \right)}{e^{-j\Omega/2} \left( e^{j\Omega/2} - e^{-j\Omega/2} \right)} \]

\[ = e^{j\Omega/2} \frac{\sin\left(\Omega(2M+1)/2\right)}{\sin(\Omega/2)} \]

L’Hôpital’s Rule

\[ \lim_{\Omega \to 0,\pm2\pi,\pm4\pi,\ldots} \frac{\sin\left(\Omega(2M+1)/2\right)}{\sin(\Omega/2)} = 2M + 1; \]

With understanding that \( X(e^{j\Omega}) \) for \( \Omega \neq 0, \pm 2\pi, \pm 4\pi, \) is obtained as limit.
Example 3.19 Inverse DTFT of A Rectangular Pulse

\[ X(e^{j\Omega}) = \begin{cases} 
1, & |\Omega| \leq W \\
0, & W < |n| < \pi 
\end{cases} \]

\[ x[n] = \frac{1}{2\pi} \int_{-W}^{W} e^{j\Omega n} d\Omega \]

\[ x[n] = \frac{W}{\pi} \sin \left(\frac{Wn}{\pi}\right) \]
Example 3.20 DTFT of The Unit Impulse

Find the DTFT of $x[n] = \delta[n]$.

<Sol.>

1. DTFT of $x[n]$:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = 1$$

2. This DTFT pair is depicted in Fig. 3.32.

Figure 3.32 (p. 235) Example 3.20. (a) Unit impulse in the time domain. (b) DTFT of unit impulse in the frequency domain.
Example 3.21 *Inverse DTFT of A Unit Impulse Spectrum*

1. **Inverse DTFT of** $X(e^{j\Omega})$: 

   $$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\Omega) e^{j\Omega n} d\Omega.$$ 

   $$\frac{1}{2\pi} \xrightarrow{\text{DTFT}} \delta(\Omega), \quad -\pi < \Omega \leq \pi.$$ 

   ✤ We can define $X(e^{j\Omega})$ over all $\Omega$ by writing it as an infinite sum of delta functions shifted by integer multiples of $2\pi$. 

2. This DTFT pair is depicted in **Fig. 3.33**.

   ![Diagram](image.png)

   ![Diagram](image.png)

   ✤ **Dilemma:** The DTFT of $x[n] = 1/(2\pi)$ does not converge, since it is not a square summable signal, yet $x[n]$ is a valid inverse DTFT!
7. Continuous-Time Nonperiodic Signals—The Fourier Transform

1. The Fourier transform (FT) is used to represent a continuous-time nonperiodic signal as a superposition of complex sinusoids.

2. FT representation of time-domain signal $x(t)$:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \quad (3.35)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad (3.36)$$

Frequency-domain representation of the signal $x(t)$

3. Notation for FT pair:

$$x(t) \underset{\text{FT}}{\longleftrightarrow} X(j\omega).$$
7. Continuous-Time Nonperiodic Signals—The Fourier Transform

Convergence condition for FT:
1) Approximation:

\[
\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega,
\]

Squared error between \(x(t)\) and \(\hat{x}(t)\): the error energy, given by

\[
\int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 dt,
\]

is zero if \(x(t)\) is square integrable, i.e., if

\[
\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty.
\]

Zero squared error does not imply pointwise convergence [i.e., \(x(t) = \hat{x}(t)\)] at all values of \(t\).

There is zero energy in the difference of \(x(t)\) and \(\hat{x}(t)\).

Pointwise convergence of \(x(t) = \hat{x}(t)\) is guaranteed at all values of \(t\) except those corresponding to discontinuities if \(x(t)\) satisfies the Dirichlet conditions for nonperiodic signals:

1. \(x(t)\) is absolutely integrable:

\[
\int_{-\infty}^{\infty} |x(t)| dt < \infty.
\]

2. \(x(t)\) has a finite number of maximum, minima, and discontinuities in any finite interval.

3. The size of each discontinuity is finite.
Example 3.25 FT of A Rectangular Pulse
Consider the rectangular pulse depicted in Fig. 3.40 (a) and defined as

\[
x(t) = \begin{cases} 
1, & -T_0 < t < T_0 \\
0, & |t| > T_0 
\end{cases}
\]

Find the FT of \( x(t) \).

1. The rectangular pulse \( x(t) \) is absolutely integrable, provided that \( T_0 < \infty \).

2. FT of \( x(t) \):

\[
X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt = \int_{-T_0}^{T_0} e^{-j\omega t} \, dt \\
= -\frac{1}{j\omega} e^{-j\omega T_0} \bigg|_{-T_0}^{T_0} = \frac{2}{\omega} \sin(\omega T_0), \quad \omega \neq 0
\]

3. For \( \omega = 0 \), the integral simplifies to \( 2T_0 \).

\[
\lim_{\omega \to 0} \frac{2}{\omega} \sin(\omega T_0) = 2T_0.
\]

\[
X(j\omega) = \frac{2}{\omega} \sin(\omega T_0),
\]

With understanding that the value at \( \omega = 0 \) is obtained by evaluating a limit.
7. Continuous-Time Nonperiodic Signals—The Fourier Transform

4. Magnitude spectrum:

\[ |X(j\omega)| = 2 \left| \frac{\sin(\omega T_0)}{\omega} \right|, \]

5. Phase spectrum:

\[ \text{arg}\{X(j\omega)\} = \begin{cases} 0, & \sin(\omega T_0)/\omega > 0 \\ \pi, & \sin(\omega T_0)/\omega < 0 \end{cases} \]

6. \(X(j\omega)\) in terms of sinc function:

\[ X(j\omega) = 2T_0 \text{sinc}(\omega T_0/\pi). \]

As \(T_0\) increases, the nonzero time extent of \(x(t)\) increases, while \(X(j\omega)\) becomes more concentrated about the frequency origin.
Example 3.26 **Inverse FT of A Rectangular Pulse**
Find the inverse FT of the rectangular spectrum depicted in Fig. 3. 42 (a) and given by

\[
X(j\omega) = \begin{cases} 
1, & -W < \omega < W \\
0, & |\omega| > W 
\end{cases}
\]

**<Sol.>**

Figure 3.42 (p. 246)

Example 3.26. (a) Rectangular spectrum in the frequency domain.
(b) Inverse FT in the time domain.

1. Inverse FT of \(x(t)\):

\[
x(t) = \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} d\omega = -\frac{1}{j\pi t} e^{j\omega t} \bigg|_{-W}^{W} = \frac{1}{\pi t} \sin(Wt), \quad t \neq 0
\]
7. Continuous-Time Nonperiodic Signals—The Fourier Transform

2. When \( t = 0 \), the integral simplifies to \( W/\pi \), i.e.,

\[
\lim_{t \to 0} \frac{1}{\pi t} \sin(Wt) = \frac{W}{\pi},
\]

3. Inverse FT is usually written as

\[
x(t) = \frac{1}{\pi t} \sin(Wt),
\]
or

\[
x(t) = \frac{W}{\pi} \text{sinc}\left(\frac{Wt}{\pi}\right),
\]

With understanding that the value at \( t = 0 \) is obtained as a limit.

As \( W \) increases, the frequency-domain representation becomes less concentrated about \( \omega = 0 \), while the time-domain representation \( X(j\omega) \) becomes more concentrated about \( t = 0 \).
7. Continuous-Time Nonperiodic Signals – The Fourier Transform

Example 3.27 **FT of The Unit Impulse**
Find the FT of \( x(t) = \delta(t) \).

\(<\text{Sol.}>\)

1. \( x(t) \) does not satisfy the Dirichlet conditions, since the discontinuity at the origin is infinite.

2. FT of \( x(t) \):

\[
X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1
\]

\( \delta(t) \xrightarrow{FT} 1 \).

Example 3.28 **Inverse FT of An Impulse Spectrum**
Find the inverse FT of \( X(j\omega) = 2\pi \delta(\omega) \).

\(<\text{Sol.}>\)

1. Inverse FT of \( X(j\omega) \):

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega) e^{j\omega t} d\omega = 1
\]

2. Notation of FT pair:

\( 1 \xrightarrow{FT} 2\pi \delta(\omega) \)

Duality between Example 3.27 and 3.28
# 8. Properties of Fourier Representation

| **Table 3.2** The Four Fourier Representations |
|---|---|---|
| **Time Domain** | **Periodic** $(t, n)$ | **Non-periodic** $(t, n)$ |
| **Continuous** $(t)$ | **Fourier Series** |
|  | $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$ |
|  | $X[k] = \frac{1}{T} \int_{0}^{T} x(t) e^{-jk\omega_0 t} dt$ |
|  | $x(t)$ has period $T$, $\omega_0 = 2\pi / T$ |
| **Discrete** $[n]$ | **Fourier Series** |
|  | $x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}$ |
|  | $X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{jk\Omega_0 n}$ |
|  | $x[n]$ and $X[k]$ have period $N$, $\Omega_0 = 2\pi / N$ |
| **Continuous** $(\omega, \Omega)$ | **Fourier Transform** |
|  | $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$ |
|  | $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ |
| **Frequency Domain** | **Nonperiodic** $(k, \omega)$ |
|  | **Discrete-Time Fourier Transform** |
|  | $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$ |
|  | $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$ |
|  | $X(e^{j\Omega})$ has period $2\pi$ |
8. Properties of Fourier Representation

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} \, d\omega \]

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \]

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n} \, d\Omega \]

\[ X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \]

\[ X(e^{j\Omega}) \text{ has period } 2\pi \]

\[ x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \]

\[ X[k] = \frac{1}{T} \int_{0}^{T} x(t)e^{-jk\omega_0 t} \, dt \]

\[ x(t) \text{ has period } T, \quad \omega_0 = \frac{2\pi}{T} \]

\[ x[n] = \sum_{k=0}^{N-1} X[k]e^{j\Omega_0 n} \]

\[ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]e^{j\Omega_0 n} \]

\[ x[n] \text{ and } X[k] \text{ have period } N, \quad \Omega_0 = \frac{2\pi}{N} \]
8. Properties of Fourier Representation

- Periodic ↔ Discrete

<table>
<thead>
<tr>
<th>Time-Domain Property</th>
<th>Frequency-Domain Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous</td>
<td>Nonperiodic</td>
</tr>
<tr>
<td>Discrete</td>
<td>Periodic</td>
</tr>
<tr>
<td>Periodic</td>
<td>Discrete</td>
</tr>
<tr>
<td>Nonperiodic</td>
<td>Continuous</td>
</tr>
</tbody>
</table>
9. Linearity and Symmetry Properties

- Linearity property for four Fourier representations

\[
\begin{align*}
z(t) &= ax(t) + by(t) \quad \xrightarrow{\text{FT}} \quad Z(j\omega) = aX(j\omega) + bY(j\omega) \\
z(t) &= ax(t) + by(t) \quad \xrightarrow{\text{FS; } \omega_o} \quad Z[k] = aX[k] + bY[k] \\
z[n] &= ax[n] + by[n] \quad \xrightarrow{\text{DTFT}} \quad Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega}) \\
z[n] &= ax[n] + by[n] \quad \xrightarrow{\text{DTFS; } \Omega_o} \quad Z[k] = aX[k] + bY[k]
\end{align*}
\]
9. Linearity and Symmetry Properties

1. Signal $z(t)$:

$$z(t) = \frac{3}{2} x(t) + \frac{1}{2} y(t)$$

where $x(t)$ and $y(t)$

2. $X[k]$ and $Y[k]$:

$$x(t) \overset{FS;2\pi}{\leftrightarrow} X[k] = \left(\frac{1}{k\pi}\right)\sin\left(k\pi/4\right)$$

$$y(t) \overset{FS;2\pi}{\leftrightarrow} Y[k] = \left(\frac{1}{k\pi}\right)\sin\left(k\pi/2\right)$$

3. FS of $z(t)$:

$$z(t) \overset{FS;2\pi}{\leftrightarrow} Z[k] = \left(\frac{3}{2k\pi}\right)\sin\left(k\pi/2\right) + \left(\frac{1}{2k\pi}\right)\sin\left(k\pi/2\right)$$
9. Linearity and Symmetry Properties

Symmetry Properties: Real and Imaginary Signals

- Symmetry property for real-valued signal \( x(t) \):

1. FT of \( x(t) \):

\[
X^*(j\omega) = \left[ \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \right]^* = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t} dt \quad (3.37)
\]

2. Since \( x(t) \) is real valued, \( x(t) = x^*(t) \). Eq. (3.37) becomes

\[
X^*(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t} dt \quad (3.38)
\]

\( X(j\omega) \) is complex-conjugate symmetric

\[
X^*(j\omega) = X(-j\omega)
\]

\[
\text{Re}\{X(j\omega)\} = \text{Re}\{X(-j\omega)\} \quad \text{and} \quad \text{Im}\{X(j\omega)\} = -\text{Im}\{X(-j\omega)\}
\]
The conjugate symmetry property for DTFS:

\[ X^*[k] = X[N - k] \]

Because the DTFS coefficients are \( N \) periodic, and thus

\[ X[-k] = X[N - k] \]

The symmetry conditions in all four Fourier representations of real-valued signals are indicated in Table 3.4.

Table 3.4  **Symmetry Properties for Fourier Representation of Real- and Imaginary-Valued Signals**

<table>
<thead>
<tr>
<th>Representation</th>
<th>Real-Valued Time Signals</th>
<th>Imaginary-Valued Time Signals</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FT</strong></td>
<td>( X^*(j\omega) = X(-j\omega) )</td>
<td>( X^*(j\omega) = -X(-j\omega) )</td>
</tr>
<tr>
<td><strong>FS</strong></td>
<td>( X^*[k] = X[-k] )</td>
<td>( X^*[k] = -X[-k] )</td>
</tr>
<tr>
<td><strong>DTFT</strong></td>
<td>( X^*(e^{j\Omega}) = X(e^{-j\Omega}) )</td>
<td>( X^*(e^{j\Omega}) = -X(e^{-j\Omega}) )</td>
</tr>
<tr>
<td><strong>DTFS</strong></td>
<td>( X^*[k] = X[-k] )</td>
<td>( X^*[k] = -X[-k] )</td>
</tr>
</tbody>
</table>
9. Linearity and Symmetry Properties

- Continuous-time case:

1. Input signal of LTI system:

\[ x(t) = A \cos(\omega t - \phi) \]

2. Real-valued impulse response of LTI system is denoted by \[ h(t) \].

3. Output signal of LTI system:

\[ y(t) = |H(j\omega)| \frac{A}{2} e^{j(\omega t - \phi + \text{arg}(H(j\omega))]} + |H(-j\omega)| \frac{A}{2} e^{-j(\omega t - \phi - \text{arg}(H(j\omega))]} \]

Exploiting the symmetry conditions:

- \[ |H(j\omega)| = |H(-j\omega)| \]
- \[ \text{arg}(H(j\omega)) = -\text{arg}(H(-j\omega)) \]

\[ y(t) = |H(j\omega)| A \cos(\omega t - \phi + \text{arg}(H(j\omega))] \]
9. Linearity and Symmetry Properties

- Discrete-time case: 
  \[ x[n] = A \cos(\Omega n - \phi) \]

2. Real-valued impulse response of LTI system is denoted by \( h[n] \).
3. Output signal of LTI system:
   \[ y[n] = |H(e^{j\Omega})| A \cos(\Omega n - \phi + \arg\{H(e^{j\Omega})\}) \]

- The LTI system modifies the amplitude of the input sinusoid by \( |H(e^{j\Omega})| \) and the phase by \( \arg\{H(e^{j\Omega})\} \).
- \( x(t) \) is purely imaginary:
  1. \( x^*(t) = -x(t) \).
  2. Eq.(3.37) becomes
   \[ X^*(j\omega) = -\int_{-\infty}^{\infty} x(t) e^{-j(\omega)t} dt \]
   \[ X^*(j\omega) = -X(-j\omega) \] (3.39)

- \[ \text{Re}\{X(j\omega)\} = -\text{Re}\{X(-j\omega)\} \quad \text{and} \quad \text{Im}\{X(j\omega)\} = \text{Im}\{X(-j\omega)\} \]
9. Linearity and Symmetry Properties

- **Symmetry Properties: Even and Odd Signals**

1. \(x(t)\) is real valued and has even symmetry.
   
   \[x^*(t) = x(t)\text{ and } x(-t) = x(t)\]

2. Eq.(3.37) becomes
   
   \[X^*(j\omega) = \int_{-\infty}^{\infty} x(-t)e^{-j\omega(-t)}dt\]
   
   Change of variable \(\tau = -t\)

3. Conclusion:
   
   1) The imaginary part of \(X(j\omega) = 0\):
      
      \[X^*(j\omega) = X(j\omega)\]
      
      If \(x(t)\) is real and even, then \(X(j\omega)\) is real.

   2) If \(x(t)\) is real and odd, then \(X^*(j\omega) = -X(j\omega)\) and \(X(j\omega)\) is imaginary.
10. Convolution Property

**Nonperiodic Signals**

1. Convolution of two nonperiodic continuous-time signals $x(t)$ and $h(t)$:

\[
y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau
\]

2. FT of $x(t-\tau)$:

\[
x(t-\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-\tau)}d\omega
\]

3. Substituting this expression into the convolution integral yields

\[
y(t) = \int_{-\infty}^{\infty} h(\tau)\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}e^{-j\omega\tau}d\omega\right]d\tau
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau X(j\omega)e^{j\omega t}d\omega
\]

\[
y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega
\]

and we identify $H(j\omega)X(j\omega)$ as the FT of $y(t)$.

\[
y(t) = h(t) * x(t) \quad \overset{FT}{\rightarrow} \quad Y(j\omega) = X(j\omega)H(j\omega)
\]

(3.40)
10. Convolution Property

Example

\[ x(t) \xrightarrow{FT} X(j\omega) = \frac{4}{\omega^2} \sin^2(\omega) \]

\[ z(t) = \begin{cases} 
1, & |t| < 1 \\
0, & |t| > 1
\end{cases} \xrightarrow{FT} Z(j\omega) \]

\[ x(t) = z(t) * z(t) \]
10. Convolution Property

**Convolution of Nonperiodic Signals — Discrete-time case**

If

\[ x[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\Omega}) \quad \text{and} \quad h[n] \overset{DTFT}{\longleftrightarrow} H(e^{j\Omega}) \]

\[ y[n] = x[n] * h[n] \overset{DTFT}{\longleftrightarrow} Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) \]  

(3.41)
10.2 Filtering

1. Multiplication in frequency domain $\leftrightarrow$ **Filtering**.

2. The terms “filtering” implies that some frequency components of the input are eliminated while others are passed by the system unchanged.

3. System Types of filtering:
   1) Low-pass filter
   2) High-pass filter
   3) Band-pass filter

4. Realistic filter:
   1) Gradual transition band
   2) Nonzero gain of stop band

5. Magnitude response of filter:
   
   $20 \log |H(j\omega)|$ or $20 \log |H(e^{j\Omega})|$ [dB]

The characterization of DT filter is based on its behavior in the frequency range $\pi < \Omega \leq \pi$ because its frequency response is $2\pi$-periodic.
10.2 Filtering
Example 3.33 **RC Circuit: Filtering**

For the RC circuit depicted in **Fig. 3.54**, the impulse response for the case where $y_C(t)$ is the output is given by

$$h_C(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

Since $y_R(t) = x(t) - y_C(t)$, the impulse response for the case where $y_R(t)$ is the output is given by

$$h_R(t) = \delta(t) - \frac{1}{RC} e^{-t/RC} u(t)$$

**Figure 3.54** (p. 264) - RC circuit with input $x(t)$ and outputs $y_C(t)$ and $y_R(t)$.

$$H_C(j\omega) = \frac{1}{i\omega RC + 1}$$

$$H_R(j\omega) = \frac{i\omega RC}{i\omega RC + 1}$$
10. Convolution Property

- **Frequency response of a system**
  - The ratio of the FT or DTFT of the output to the input.

  For CT system:
  \[
  H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \quad (3.42)
  \]

  For DT system:
  \[
  H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} \quad (3.43)
  \]
10. Convolution Property

- **Recover the input of the system from the output**

\[
X(j\omega) = H^{\text{inv}}(j\omega)Y(j\omega) \quad \text{CT case}
\]

where

\[
H^{\text{inv}}(j\omega) = 1/H(j\omega)
\]

- **Equalizer**

  - The process of recovering the input from the output is known as equalization

\[
X(e^{j\Omega}) = H^{\text{inv}}(e^{j\Omega})Y(e^{j\Omega}) \quad \text{DT case}
\]

and

\[
H^{\text{inv}}(e^{j\Omega}) = 1/H(e^{j\Omega})
\]
10. Convolution Property

**Convolution of Periodic Signals**

Define the periodic convolution of two CT signals $x(t)$ and $z(t)$, each having period $T$, as

$$y(t) = x(t) \# z(t) = \int_{0}^{T} x(\tau) z(t-\tau) d\tau$$

where the symbol $\#$ denotes that integration is performed over a single period of the signals involved.

$$y(t) = x(t) \# z(t) \quad \xrightarrow{FS; \frac{2\pi}{T}} \quad Y[k] = TX[k]Z[k]$$
10. Convolution Property

- **Discrete-Time Convolution of Periodic Sequences**
  - Define the discrete-time convolution of two $N$-periodic sequences $x[n]$ and $z[n]$

$$ y[n] = x[n] \# z[n] = \sum_{k=0}^{N-1} x[k] z[n-k] $$

**DTFS of $y[n]$**

$$ y[n] = x[n] \# z[n] \quad \xrightarrow{DTFS; \frac{2\pi}{N}} \quad Y[k] = NX[k]Z[k] $$

---

**Table 3.5 Convolution Properties**

- $x(t) \star z(t) \quad \xrightarrow{FT} \quad X(j\omega)Z(j\omega)$
- $x(t) \# z(t) \quad \xrightarrow{FS; \omega_0} \quad TX[k]Z[k]$
- $x[n] \star z[n] \quad \xrightarrow{FT} \quad X(e^{j\Omega})Z(e^{j\Omega})$
- $x[n] \# z[n] \quad \xrightarrow{DTFS; \Omega_0} \quad NX[k]Z[k]$
11. Differentiation and Integration Properties

**Differentiation in Time**

1. A nonperiodic signal $x(t)$ and its FT, $X(j\omega)$, is related by

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega
\]

Differentiating both sides with respect to $t$

\[
\frac{d}{dt} x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)j\omega e^{j\omega t} d\omega
\]

**Differentiation of** $x(t)$ **in Time-Domain $\leftrightarrow (j\omega) \times X(j\omega)$ in Frequency-Domain**

\[
F \left[ \frac{d}{dt} x(t) \right]_{\omega=0} = (j\omega) \times X(j\omega)_{\omega=0} = 0
\]
11. Differentiation and Integration Properties

- Differentiation in Frequency

1. FT of signal $x(t)$:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Differentiating both sides with respect to $\omega$ gives:

$$\frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{\infty} -jtx(t)e^{-j\omega t} dt$$

2. Differentiation property in frequency domain:

$$-jtx(t) \iff FT \iff \frac{d}{d\omega} X(j\omega)$$

◆ Differentiation of $X(j\omega)$ in Frequency-Domain $\iff (-jt) \times x(t)$ in Time-Domain
11. Differentiation and Integration Properties

Integration

Integrating nonperiodic signals

1. Define

\[ y(t) = \int_{-\infty}^{t} x(\tau) d\tau \]

(3.51)

2. By differentiation property, we have

\[ Y(j\omega) = \frac{1}{j\omega} X(j\omega) \]

(3.52)

Indeterminate at \( \omega = 0 \)

3. When the average value of \( x(t) \) is not zero, then it is possible that \( y(t) \) is not square integrable.

\[ \text{FT of } y(t) \text{ may not converge!} \]

We can get around this problem by including impulse in the transform, i.e.

\[ Y(j\omega) = \frac{1}{j\omega} X(j\omega) + c\delta(\omega) \]

\( c \) can be determined by the average value of \( x(t) \)

\[ C = \pi X(j0) \]
# 11. Differentiation and Integration Properties

## Table 3.6 Commonly Used Differentiation and Integration Properties.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Forward Transform (FT)</th>
<th>Property</th>
<th>Rearrange</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d}{dt} x(t)$</td>
<td>$\leftrightarrow^{FT}$</td>
<td>$j\omega X(j\omega)$</td>
<td>$FT$</td>
</tr>
<tr>
<td>$\frac{d}{dt} x(t)$</td>
<td>$\leftrightarrow^{FS: \omega_0}$</td>
<td>$jk\omega_0 X[k]$</td>
<td>$FS$</td>
</tr>
<tr>
<td>$-jtx(t)$</td>
<td>$\leftrightarrow^{FT}$</td>
<td>$\frac{d}{d\omega} X(j\omega)$</td>
<td>$FT$</td>
</tr>
<tr>
<td>$-jnx[n]$</td>
<td>$\leftrightarrow^{DTFT}$</td>
<td>$\frac{d}{d\Omega} X(e^{j\Omega})$</td>
<td>$DTFT$</td>
</tr>
<tr>
<td>$\int_{-\infty}^{t} x(\tau)d\tau$</td>
<td>$\leftrightarrow^{FT}$</td>
<td>$\frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega)$</td>
<td>$FT$</td>
</tr>
</tbody>
</table>
12. Time- and Frequency-Shift Properties

**Time-Shift Property**

1. Let \( z(t) = x(t - t_0) \) be a time-shifted version of \( x(t) \).
2. FT of \( z(t) \):

\[
Z(j\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} \, dt
\]

3. Change variable by \( \tau = t - t_0 \):

\[
Z(j\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_0)} \, d\tau = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \tau} \, d\tau = e^{-j\omega t_0} X(j\omega)
\]

◆ Time-shifting of \( x(t) \) by \( t_0 \) in Time-Domain

\( \leftrightarrow (e^{-j\omega t_0}) \times X(j\omega) \) in Frequency-Domain

\[
|Z(j\omega)| = |X(j\omega)| \quad \text{and} \quad \arg\{Z(j\omega)\} = \arg\{X(j\omega)\} - \omega_0 t
\]
12. Time- and Frequency-Shift Properties

<table>
<thead>
<tr>
<th>Function</th>
<th>Transformation</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(t - t_0)$</td>
<td>$\text{FT}$</td>
<td>$e^{-j\omega t_0} X(j\omega)$</td>
</tr>
<tr>
<td>$x(t - t_0)$</td>
<td>$\text{FT}$; $\omega_0$</td>
<td>$e^{-jk\omega t_0} X(k)$</td>
</tr>
<tr>
<td>$x[n - n_0]$</td>
<td>$\text{DTFT}$</td>
<td>$e^{-j\Omega n_0} X(e^{j\Omega})$</td>
</tr>
<tr>
<td>$x[n - n_0]$</td>
<td>$\text{DTFS}; \Omega_0$</td>
<td>$e^{-jk\Omega_0 n_0} X[k]$</td>
</tr>
</tbody>
</table>
12. Time- and Frequency-Shift Properties

**Frequency Shift**

1. Suppose that:

\[ x(t) \xrightarrow{FT} X(j\omega) \quad \text{and} \quad z(t) \xrightarrow{FT} Z(j\omega) \]

2. By the definition of the inverse FT, we have

\[
 z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j(\omega - \gamma)) e^{j\omega t} d\omega
\]

Substituting variables \( \eta = \omega - \gamma \) into above Eq. gives

\[
 z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\eta) e^{j(\eta+\gamma)t} d\eta = e^{j\gamma} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\eta)e^{j\eta t} d\eta = e^{j\gamma} x(t)
\]

◆ Frequency-shifting of \( X(j\omega) \) by \( \gamma \) in Frequency-Domain [i.e. \( X(j(\omega - \gamma)) \)]

\( \leftrightarrow (e^{-\gamma t}) \times x(t) \) in Time-Domain
12. Time- and Frequency-Shift Properties

\[ e^{j\gamma t} x(t) \quad \overset{FT}{\leftrightarrow} \quad X(j(\omega - \gamma)) \]
\[ e^{jk_0\omega_0 t} x(t) \quad \overset{FS; \omega_0}{\leftrightarrow} \quad x[k - k_0] \]
\[ e^{j\Gamma n} x[n] \quad \overset{DTFT}{\leftrightarrow} \quad X \left[ e^{j(\Omega - \Gamma)} \right] \]
\[ e^{jk_0\Omega_0 n} x[n] \quad \overset{FS; \Omega_0}{\leftrightarrow} \quad X[k - k_0] \]

1. FT $X(j\omega)$ in terms of a ratio of polynomials in $j\omega$:

$$X(j\omega) = \frac{b_M(j\omega)^M + \cdots + b_1(j\omega) + b_0}{(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_1(j\omega) + a_0} = \frac{B(j\omega)}{A(j\omega)}$$

Assume that $M < N$. If $M \geq N$, then we may use long division to express $X(j\omega)$ in the form

$$X(j\omega) = \sum_{k=0}^{M-N} f_k(j\omega^k) + \frac{B(j\omega)}{A(j\omega)}$$

Partial-fraction expansion is applied to this term

2. Let the roots of the denominator $A(j\omega)$ be $d_k$, $k = 1, 2, \ldots, N$. These roots are found by replacing $j\omega$ with a generic variable $v$ and determining the roots of the polynomial

$$v^N + a_{N-1}v^{N-1} + \cdots + a_1v + a_0 = 0$$

Applying the differentiation property and the pair $\delta(t) \overset{FT}{\longleftrightarrow} 1$ to these terms
13. Finding Inverse Fourier Transforms by Using Partial-Fraction Expansions

3. For $M < N$, we may write:

$$X(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\prod_{k=1}^{N} (j\omega - d_k)}$$

Assuming that all the roots $d_k$, $k = 1, 2, \ldots, N$, are distinct, we may write:

$$X(j\omega) = \sum_{k=1}^{N} \frac{C_k}{j\omega - d_k}$$

$C_k$, $k = 1, 2, \ldots, N$ are determined by the method of residues

In Example 3.24, we derived the transform pair:

$$e^{dt} u(t) \xrightarrow{FT} \frac{1}{j\omega - d}$$

This pair is valid even if $d$ is complex, provided that $\text{Re}\{d\} < 0$.

Assuming that the real part of each $d_k$, $k = 1, 2, \ldots, N$, is negative, we use linearity to write:

$$x(t) = \sum_{k=1}^{N} C_k e^{d_k t} u(t) \xrightarrow{FT} X(j\omega) = \sum_{k=1}^{N} \frac{C_k}{j\omega - d_k}$$
13. Finding Inverse Fourier Transforms by Using Partial-Fraction Expansions

**Inverse Discrete-Time Fourier Transform**

1. Suppose $X(e^{j\Omega})$ is given by a ratio of polynomial in $e^{j\Omega}$, i.e.
   \[
   X(e^{j\Omega}) = \frac{\beta_M e^{-j\Omega M} + \cdots + \beta_1 e^{-j\Omega} + \beta_0}{\alpha_N e^{-j\Omega N} + \alpha_{N-1} e^{-j\Omega (N-1)} + \cdots + \alpha_1 e^{-j\Omega} + 1}
   \]
   **Normalized to 1**

2. Factor the denominator polynomial as
   \[
   \alpha_N e^{-j\Omega N} + \alpha_{N-1} e^{-j\Omega (N-1)} + \cdots + \alpha_1 e^{-j\Omega} + 1 = \prod_{k=1}^{N} (1 - d_k e^{-j\Omega})
   \]
   Replace $e^{j\Omega}$ with the generic variable $v$:
   \[
   v^N + \alpha_1 v^{N-1} + \alpha_2 v^{N-2} + \cdots + \alpha_{N-1} v + \alpha_N = 0
   \]
   **Find $d_k$, the roots of this polynomial**

3. Partial-fraction expansion:
   Assuming that $M < N$ and all the $d_k$ are distinct, we may express $X(e^{j\Omega})$ as
   \[
   X(e^{j\Omega}) = \sum_{k=1}^{N} \frac{C_k}{1 - d_k e^{-j\Omega}}
   \]
Since

\[(d_k)^n u[n] \quad \overset{DTFT}{\longleftrightarrow} \quad \frac{1}{1 - d_k e^{-j\Omega}}\]

The linearity property implies that

\[x[n] = \sum_{k=1}^{N} C_k (d_k)^n u[n]\]

Expansions for repeated roots are treated in Appendix B.
14. Multiplication Property

- Non-periodic continuous-time signals

1. Non-periodic signals: \( x(t), z(t), \) and \( y(t) = x(t)z(t) \).

2. FT of \( x(t) \) and \( z(t) \):

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jv)e^{jvt} \, dv
\]

\[
z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(j\eta)e^{j\eta t} \, d\eta
\]

\[
y(t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(jv)Z(j\eta)e^{j(\eta+v)t} \, d\eta \, dv
\]

Change variable: \( \eta = \omega - v \)

Inner Part: \( Z(j\omega) \ast X(j\omega) \)

3. FT of \( y(t) \):

\[
y(t) = x(t)z(t) \quad \xrightarrow{FT} \quad Y(j\omega) = \frac{1}{2\pi} X(j\omega) \ast Z(j\omega)
\]

Scaled by \( 1/2\pi \)

Find the FT of \( y(t) \).
14. Multiplication Property

where

\[ X(j\omega) * Z(j\omega) = \int_{-\infty}^{\infty} X(jv)Z(j(\omega - v))dv \]

◆ Multiplication of two signals in Time-Domain
  ↔ Convolution in Frequency-Domain \( \times \ (1/2\pi) \)

◆ Non-periodic continuous-time signals

1. Non-periodic DT signals: \( x[n], z[n], \) and \( y[n] = x[n]z[n] \).

2. DTFT of \( y[n] \):

\[
y[n] = x[n]z[n] \quad \overset{\text{DTFT}}{\longrightarrow} \quad Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \# Z(e^{j\Omega})
\]

where the symbol \( \# \) denotes periodic convolution.

Here, \( X(e^{j\Omega}) \) and \( X(e^{j\Omega}) \) are \( 2\pi \)-periodic, so we evaluate the convolution over a \( 2\pi \) interval:

\[
X(j\omega) \# Z(j\omega) = \int_{-\infty}^{\infty} X(e^{j\theta})Z(e^{j(\Omega-\theta)})d\theta
\]

◆ Multiplication of two signals in Time-Domain
  ↔ Convolution in Frequency-Domain \( \times \ (1/2\pi) \)
14. Multiplication Property

- Multiplication property can be used to study the effects of truncating a time-domain signal on its frequency-domain. Windowing!

- Truncate signal $x(t)$ by a window function $w(t)$ is represented by $y(t) = x(t)w(t)$

Time Interval $-T_0 < t < T_0$

Fig. 3. of (a)
14. Multiplication Property

1. FT of $y(t)$:

$$y(t) \overset{FT}{\leftrightarrow} Y(j\omega) = \frac{1}{2\pi} X(j\omega) \ast W(j\omega)$$

2. If $w(t)$ is the rectangular window depicted in Fig. 3.65 (b), we have

$$W(j\omega) = \frac{2}{\omega} \sin(\omega T_0)$$

Smoothing the details in $X(j\omega)$ and introducing oscillation near discontinuities in $X(j\omega)$.

Figure 3.65b (p. 293)

(b) Convolution of the signal and window FT’s resulting from truncation in time.
Example 3.46 Truncating the Impulse Response

The frequency response $H(e^{j\Omega})$ of an ideal discrete-time system is depicted in Fig. 3. 66 (a). Describe the frequency response of a system whose impulse response is the ideal system impulse response truncated to the interval $-M \leq n \leq M$.

<Sol.>

1. The ideal impulse response is the inverse DTFT of $H(e^{j\Omega})$.

   Using the result of Example 3.19, we write

   $$h[n] = \frac{1}{\pi n} \sin\left(\frac{\pi n}{2}\right)$$

2. Let $h_t[n]$ be the truncated impulse response:

   $$h_t[n] = \begin{cases} h[n], & |n| \leq M \\ 0, & \text{otherwise} \end{cases}$$

   $$h_t[n] = h[n]w[n]$$

   $$w[n] = \begin{cases} 1, & |n| \leq M \\ 0, & \text{otherwise} \end{cases}$$
3. Let \( h_t[n] \xrightarrow{DTFT} H_t(e^{j\Omega}) \)

Using the multiplication property in Eq. (3.57), we have

\[
H_t(e^{j\Omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\theta}) W(e^{j(\theta-\Omega)}) d\theta
\]

4. Since

\[
H[e^{j\theta}] = \begin{cases} 
1, & |\theta| < \pi/2 \\
0, & \pi/2 < |\theta| < \pi 
\end{cases}
\]

and

\[
W(e^{j(\Omega-\theta)}) = \frac{\sin((\Omega-\theta)(2M+1)/2)}{\sin((\Omega-\theta)/2)}
\]

where we have defined

\[
F_\Omega(\theta) = H(e^{j\theta})W(e^{j(\Omega-\theta)}) = \begin{cases} 
W(e^{j(\Omega-\theta)}), & |\theta| < \pi/2 \\
0, & |\theta| > \pi/2 
\end{cases}
\]

On the basis of Example 3.18

Discussion: refer to p. 295 in textbook.
14. Multiplication Property

Figure 3.66 (p. 294)
The effect of truncating the impulse response of a discrete-time system.
(a) Frequency response of ideal system. (b) $F_\Omega(\theta)$ for $\Omega$ near zero.
14. Multiplication Property

Figure 3.66 (p. 294)
The effect of truncating the impulse response of a discrete-time system. (c) $F_{\Omega}(\theta)$ for $\Omega$ slightly greater than $\pi/2$. (d) Frequency response of system with truncated impulse response.

$H(e^{j\Omega})$ is the area under $F_{\Omega}(\theta)$ between $\theta = -\pi/2$ and $\theta = \pi/2$. 

$v \equiv \omega, u \equiv \theta$
### Table 3.9 Multiplication Properties of Fourier Representations

<table>
<thead>
<tr>
<th>Continuous-Time Case</th>
<th>Discrete-Time Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(t)z(t)$</td>
<td>$x(t)z(t)$</td>
</tr>
<tr>
<td>$\xrightarrow{\text{FT}}$</td>
<td>$\xrightarrow{\text{FS}}$; $\omega_o$</td>
</tr>
<tr>
<td>$\frac{1}{2\pi} X(j\omega)*Z(j\omega)$</td>
<td>$X[k]*Z[k]$</td>
</tr>
<tr>
<td>$x[n]z[n]$</td>
<td>$x[n]z[n]$</td>
</tr>
<tr>
<td>$\xrightarrow{\text{DTFT}}$</td>
<td>$\xrightarrow{\text{DTFT}}$; $\omega_o$</td>
</tr>
<tr>
<td>$\frac{1}{2\pi} X(e^{j\Omega})# Z(e^{j\Omega})$</td>
<td>$X[k]# Z[k]$</td>
</tr>
</tbody>
</table>
15. Scaling Properties

1. Let \( z(t) = x(at) \).
2. FT of \( z(t) \):

\[
Z(j\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(at)e^{-j\omega t} dt
\]

Changing variable: \( \tau = at \)

\[
Z(j\omega) = \begin{cases} 
(1/a)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau} d\tau, & a > 0 \\
(1/a)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau} d\tau, & a < 0 
\end{cases}
\]

\[
Z(j\omega) = (1/|a|)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau} d\tau,
\]

\[
z(t) = x(at) \quad \overset{\text{FT}}{\rightarrow} \quad (1/|a|)X(j\omega/a).
\] (3.60)

◆ Scaling in Time-Domain  ⇔  Inverse Scaling in Frequency-Domain

Signal expansion or compression!

Refer to Fig. 3.70.
15. Scaling Properties

Figure 3.70 (p. 300)
The FT scaling property. The figure assumes that $0 < a < 1$.\[ x(t) \quad v \equiv \infty \]

The figure assumes that $0 < a < 1$.\[ x(at) \]

The figure assumes that $0 < a < 1$.\[ X(jv) \]

The figure assumes that $0 < a < 1$.\[ \frac{1}{|a|} X(jv/a) \]
Example 3.48 Scaling a Rectangular Pulse

Let the rectangular pulse

\[ x(t) = \begin{cases} 
1, & |t| < 1 \\
0, & |t| > 1 
\end{cases} \]

Use the FT of \( x(t) \) and the scaling property to find the FT of the scaled rectangular pulse

\[ y(t) = \begin{cases} 
1, & |t| < 2 \\
0, & |t| > 2 
\end{cases} \]

<Sol.>

1. Substituting \( T_0 = 1 \) into the result of Example 3.25 gives

\[ X(j\omega) = \frac{2}{\omega} \sin(\omega) \]

2. Note that \( y(t) = x(t/2) \).

\[ Y(j\omega) = 2X(j2\omega) = 2 \left( \frac{2}{2\omega} \right) \sin(2\omega) = \frac{2}{\omega} \sin(2\omega) \]

Scaling property and \( a = 1/2 \)

Fig. 3.71

\[ \text{Substituting } T_0 = 2 \text{ into the result of Example 3.25 can also give the answer!} \]
15. Scaling Properties

Figure 3.71 (p. 301)
Application of the FT scaling property in Example 3.48.
(a) Original time signal. (b) Original FT. (c) Scaled time signal $y(t) = x(t/2)$. (d) Scaled FT $Y(j\omega) = 2X(j2\omega)$.

$v \equiv \omega$
16. Parseval Relationships

- The energy or power in the time-domain representation of a signal is equal to the energy or power in the frequency-domain representation.

**Case for CT nonperiodic signal: $x(t)$**

1. Energy in $x(t)$:

$$W_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

2. Note that $|x(t)|^2 = x(t)x^*(t)$

Express $x^*(t)$ in terms of its FT $X(j\omega)$:

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t} d\omega$$

The integral inside the braces is the FT of $x(t)$. The integral of $|x(t)|^2$ is:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$
## 16. Parseval Relationships

<table>
<thead>
<tr>
<th>Representations</th>
<th>Parseval Relationships</th>
</tr>
</thead>
<tbody>
<tr>
<td>FT</td>
<td>( \int_{-\infty}^{\infty}</td>
</tr>
<tr>
<td>FS</td>
<td>( \frac{1}{T} \int_{0}^{T}</td>
</tr>
<tr>
<td>DTFT</td>
<td>( \sum_{n=-\infty}^{\infty}</td>
</tr>
<tr>
<td>DTFS</td>
<td>( \frac{1}{N} \sum_{n=0}^{N-1}</td>
</tr>
</tbody>
</table>
16. Parseval Relationships

Example 3.50 Calculating Energy in a Signal

Let

\[ x[n] = \frac{\sin(Wn)}{\pi n} \]

Use Parseval’s theorem to evaluate

\[ \chi = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} \frac{\sin^2(Wn)}{\pi^2 n^2} \]

Direct calculation in time-domain is very difficult!

<Sol.>

1. Using the DTFT Parseval relationship in Table 3.10, we have

\[ \chi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{j\Omega}) \right|^2 d\Omega \]

2. Since

\[ x[n] \overset{DTFT}{\rightarrow} X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| \leq W \\ 0, & W < |\Omega| < \pi \end{cases} \]

\[ \chi = \frac{1}{2\pi} \int_{-W}^{W} 1 d\Omega = \frac{W}{\pi} \]
17. Time-Bandwidth Product

1. Preview:

\[ x(t) = \begin{cases} 1, & |t| \leq T_0 \\ 0, & |t| > T_0 \end{cases} \]

\[ \X(j\omega) = \frac{2\sin(\omega T_0)}{\omega} \]

Fig. 3.72

Figure 3.72 (p. 305)
Rectangular pulse illustrating the inverse relationship between the time and frequency extent of a signal.

2. \( x(t) \) has time extent \( 2T_0 \). \( X(j\omega) \) is actually of infinite extent frequency.

3. The mainlobe of sinc function is fallen in the interval of \( |\omega| < \pi / T_0 \), in which contains the majority of its energy.
17. Time-Bandwidth Product

4. The product of the time extent $T_0$ and mainlobe width $2\pi/T_0$ is a constant.

5. Compressing a signal in time leads expansion in the frequency domain and vice versa.

6. Bandwidth: The extent of the signal’s significant frequency content.
   1) A mainlobe bounded by nulls. Ex. Lowpass filter → One half the width of mainlobe.
   2) The frequency at which the magnitude spectrum is $1/\sqrt{2}$ times its peak values.

7. Effective duration of signal $x(t)$:

   \[
   \begin{aligned}
   T_d &= \left[ \frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \right]^{1/2}
   \end{aligned}
   \]  
   (3.63)

8. Effective bandwidth of signal $x(t)$:

   \[
   \begin{aligned}
   B_w &= \left[ \frac{\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega} \right]^{1/2}
   \end{aligned}
   \]  
   (3.64)
9. The time-bandwidth product for any signal is lower bounded according to the relationship

\[ T_d B_w \geq 1/2 \]  \hspace{1cm} (3.65)

We cannot simultaneously decrease the duration and bandwidth of a signal.

**Uncertainty principle!**

**Example 3.51  Bounding the Bandwidth of a Rectangular Pulse**

Let

\[ x(t) = \begin{cases} 
1, & |t| \leq T_o \\
0, & |t| > T_o 
\end{cases} \]

Use the uncertainty principle to place a lower bound on the effective bandwidth of \( x(t) \).

**<Sol.>**

1. \( T_d \) of \( x(t) \):

\[ T_d = \left[ \frac{\int_{-T_o}^{T_o} t^2 \, dt}{\int_{-T_o}^{T_o} dt} \right]^{1/2} \]
17. Time-Bandwidth Product

\[ T_d = \left[ \frac{\int_{-T_o}^{T_o} t^2 dt}{\int_{-T_o}^{T_o} dt} \right]^{1/2} = \left[ \frac{1/(2T_o)}{(1/3)t^3}\big|_{-T_o}^{T_o} \right]^{1/2} = T_d / \sqrt{3} \]

2. The uncertainty principle given by Eq. (3.65) states that

\[
B_w \geq \frac{1}{2T_d} \quad \Rightarrow \quad B_w \geq \frac{\sqrt{3}}{2T_o}
\]
18. Duality

Duality of the FT

1. FT pair:

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \quad \text{and} \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \]

2. General equation:

\[ y(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(\eta)e^{j\nu \eta} d\eta \]  \hspace{1cm} (3.66)

Choose \( \nu = t \) and \( \eta = \omega \) the Eq. (3.66) implies that

\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(\omega)e^{j\omega t} d\omega \quad \overset{\text{FT}}{\longrightarrow} \quad y(t) \quad \longleftrightarrow \quad z(\omega) \] \hspace{1cm} (3.67)

3. Interchange the role of time and frequency by letting \( \nu = -\omega \) and \( \eta = t \), then Eq. (3.66) implies that

\[ y(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(t)e^{-j\omega t} dt \quad \overset{\text{FT}}{\longrightarrow} \quad z(t) \quad \longleftrightarrow \quad 2\pi y(-\omega) \] \hspace{1cm} (3.68)
18. Duality

\[ f(t) \leftrightarrow F(j\omega) \]

\[ F(jt) \leftrightarrow 2\pi f(-\omega) \]
Example

Example 3.52 Applying Duality

Find the FT of

\[ x(t) = \frac{1}{1 + jt} \]

\(<\text{Sol.}>\)

1. Note that:

\[ f(t) = e^{-t}u(t) \quad \leftrightarrow^{\text{FT}} \quad F(j\omega) = \frac{1}{1 + j\omega} \]

2. Replacing \( \omega \) by \( t \), we obtain

\[ F(jt) = \frac{1}{1 + jt} \]

3. Duality property:

\[ F(jt) \quad \leftrightarrow^{\text{FT}} \quad 2\pi f(-\omega) \]

\[ X(j\omega) = 2\pi f(-\omega) = 2\pi e^{\omega}u(-\omega) \]
Duality

Figure 3.74 (p. 309)
The FT duality property.
18. Duality

1. DTFS Pair for \( x[n] \):

\[
x[n] = \sum_{k=0}^{N-1} X[k] e^{j k \Omega_0 n}
\]

and

\[
X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j k \Omega_0 n}
\]

2. The DTFS duality is stated as follows: if

\[
x[n] \xrightarrow{\text{DTFS; } \frac{2\pi}{N}} X[k]
\]

(3.71)

\[
X[n] \xrightarrow{\text{DTFS; } \frac{2\pi}{N}} \frac{1}{N} x[-k]
\]

(3.72)

\( N = \text{time index; } k = \text{frequency index} \)
18. Duality

**DTFT**

\[ z(t) = \sum_{k=-\infty}^{\infty} Z[k]e^{jk\omega_0 t} \]

\[ X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} x[n]e^{-j\Omega n} \]

3. Duality relationship between \( z(t) \) and \( X(e^{j\Omega}) \)

Assumption: \( \omega_0 = 1 \)

\[ x[n] \xrightarrow{DTFS} X(e^{j\Omega}) \]

\[ X(e^{jt}) \xrightarrow{FS;1} x[-k] \]

<table>
<thead>
<tr>
<th>FT</th>
<th>( f(t) )</th>
<th>( F(j\omega) )</th>
<th>( F(jt) )</th>
<th>( 2\pi f(-\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DTFS</td>
<td>( x[n] )</td>
<td>( X[k] )</td>
<td>( X[n] )</td>
<td>( (1/N)x[-k] )</td>
</tr>
<tr>
<td>FS-DTFT</td>
<td>( x[n] )</td>
<td>( X(e^{j\Omega}) )</td>
<td>( X(e^{jt}) )</td>
<td>( x[-k] )</td>
</tr>
</tbody>
</table>

**Table 3.11 Duality Properties of Fourier Representations**
A.1 Wind Instruments

- Fipples, pressure-excited reeds, or buzzing lips are coupled with resonating air column.
A.2 String Instruments

- One or more stretched stringed strings are made to vibrate, typically by bowing, plunking or striking, in conjunction with resonating boxes or sound boards.
A.3 Percussion Instruments

- Include virtually any object that may be struck, typically consisting of bars, plates, or stretched membranes and associated resonators.
A.4 Cello & Clarinet
A.5 Trumpet & Bass Drum
A.6 Speech

**Models**

\[
y(n) = -\sum_{k=1}^{p} a_k y(n-k) + G \sum_{r=0}^{q} b_r x(n-r)
\]

Pitch period P

Voiced

Unvoiced

voiced/unvoiced switch

Gain estimate

\[ \hat{\theta}_o \]

\[ \hat{\theta}(z) \]

signal speech

y(n)